GROWING Mathematically

Multiplicative Thinking

Teacher's Manual

Background reading

Growing Mathematically is a project funded by the Australian Federal Government. It builds on materials that were developed as part of the RMIT Reframing Mathematical Futures II and the Scaffolding Numeracy in the Middle Years (SNMY) projects. SNMY content is currently hosted by the Victorian Department of Education.

GROWING MATHEMATICALLY: Multiplicative Thinking

TEACHER MANUAL

Supporting a targeted teaching approach to multiplicative thinking in the middle years based on an evidenced-based learning progression

This resource has been produced by the Australian Association of Mathematics Teachers (AAMT), in collaboration with Emeritus Professor Dianne Siemon of RMIT and her colleagues with funding provided by the Australian Government Department of Education.

The aim of the Manual (i.e., the Mathematics Teacher Practice Guide) is to add value to the existing formative assessment materials for multiplicative thinking developed by the *Scaffolding Numeracy in the Middle Years Project* (Siemon, Breed, Dole, Izard, & Virgona, 2006)[1].

The update has been made possible by the results of the *Reframing Mathematical Futures* (2013-2018) projects that explored the efficacy of using the SNMY materials in secondary schools alongside the development of similar formative assessment materials for algebraic, geometrical, and statistical reasoning (Siemon, Callingham, Day, Horne, Seah, Stevens, & Watson, 2018).

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What is MT? Definition of Multiplicative Thinking

Why is MT important? Brief description of the SNMY and RMFII projects, data to show MT the issue in the middle years and targeted teaching works!

Supporting the development of multiplicative thinking – the SNMY formative assessment materials

Targeted teaching – description and evidence to show that TT works

Instructions – how to administer and mark assessment options

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1 https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/scaffoldnum.aspx

Student work samples – to help interpret scoring rubrics

Links to the Learning Assessment Framework for Multiplicative Thinking 2021 – revised learning progression, teaching advice, and related resources

Case Studies – examples to show how schools have implemented a TT approach

References/Further Reading – links to relevant research papers, wider reading (to be completed)

Research Basis – Appendices (to be added)

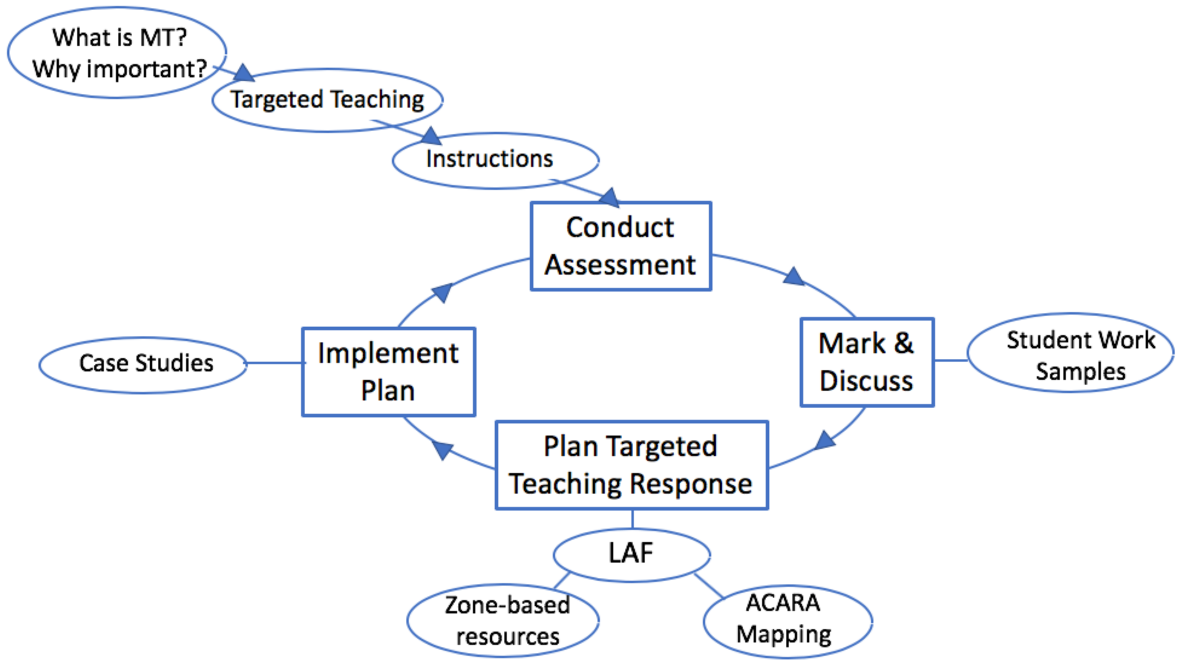


Figure 1. A conceptual map of the Growing Mathematically – Multiplicative Thinking Resource

What is Multiplicative Thinking?

Multiplicative thinking involves recognising and working with relationships between quantities. Although some aspects of multiplicative thinking are available to young children, multiplicative thinking is substantially more complex than additive thinking and may take many years to achieve (Lamon, 2012; Vergnaud, 1983). This is because multiplicative thinking is concerned with processes such as replicating, shrinking, enlarging, and exponentiating that are fundamentally more complex than the more obvious processes of aggregation and disaggregation associated with additive thinking and the use of whole numbers (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015).

Multiplicative thinking is qualitatively different to additive thinking. It is evident when students:

- work flexibly and confidently with an extended range of numbers (i.e. larger whole numbers, fractions decimals, per cent, and ratios);
- solve problems involving multiplication and division, including direct and indirect proportion using strategies appropriate to the task; and
- explain and communicate their reasoning in a variety of ways (e.g. words, diagrams, symbolic expressions, and written algorithms. (Siemon, Breed, & Virgona, 2005).

In short, where additive thinking involves the aggregation or disaggregation of collections (e.g., $634 + $478 or finding the difference between 82 kg and 67 kg), multiplicative thinking involves reasoning with relationships between quantities, for example,

> 3 bags of wool per sheep, 5 sheep, how many bags of wool?,
>
> At an average speed of 85 km/hour, how long will it take to travel 367 km?.

Additive problems generally involve one measure space (e.g., dollars or kilograms) while multiplicative problems generally involve working with two (or more) measure spaces (e.g. bags of wool, number of sheep) and a relationship between the two (i.e. 3 bags of wool per sheep).

Because simple multiplicative problems such as '24 strawberry plants per row, 17 rows, how many strawberry plants?' can be solved additively using repeated addition or by using a learnt algorithm and known facts, it can be difficult to determine whether or not a student is thinking multiplicatively. Where this becomes apparent is where the problems involve larger whole numbers, fractions, decimals, per cent or ratios, and/or more complex relationships between quantities. For example, the following problems will generally provoke a range of strategies, not all of which are multiplicative

> A muffin recipe uses 2/3 cup of milk to make 12 muffins. How many muffins can be made with 6 cups of milk?
>
> A small business owner wants to offer a further 20% discount on her summer clothing range, but she needs to ensure she covers the wholesale price. The wholesale price of a summer top was $73. If the original price of the top was $139 and it was currently on sale for 30%, can she offer a 20% discount on the already discounted price?
>
> Mobile phone covers are offered in 5 different sizes, 3 different styles, and 14 different colours. How many different phone covers need to be ordered to have 3 of each type in stock?

Sam said that doubling the dimensions of the garden box would double the volume. Is he correct? Use as much mathematics as you can to justify your conclusion.

If it takes 3 men 24 hours to paint a house, how long will it take 2 men to paint the house?

A wildlife officer estimated that there were 73 koalas in one forest reserve of 328 hectares and 62 in another forest reserve of 263 hectares. Which forest reserve provided more space for each koala?

Why is Multiplicative Thinking important?

Multiplicative thinking is crucial to success in school mathematics. It underpins nearly all of the topics considered in the middle years and beyond (see Siemon, 2013) and it is fundamental to careers in science, technology, engineering and mathematics (STEM).

Multiplicative thinking is needed to support efficient solutions to more difficult problems involving multiplication and division, fractions, decimal fractions, ratio, rates and percentage, and to solve proportional reasoning problems as they arise in algebra, geometry, measurement, statistics, and probability.

However, Australian research suggests that at least 25% and up to 55% of students in Year 8 do not have access to this critical capacity (Siemon, Breed, Dole, Izard, & Virgona, 2006; Siemon, 2013, 2016, 2019; Siemon, Banks, & Prasad, 2018).

A large-scale study involving just under 7000 Victorian students in Years 5 to 9 found that there was a seven-year range in student mathematics achievement in each year level, which was almost entirely due to the extent to which students had access to multiplicative thinking (Siemon & Virgona, 2001). More recent studies involving up to 32 secondary schools across Australia have confirmed that access to multiplicative thinking remains the reason for the significant difference in student mathematics achievement in Years 7 to 9 (e.g., Siemon, 2013, 2016, 2019; Siemon, Banks, & Prassad, 2018).

Lack of access to multiplicative thinking helps explain the reported decline in the performance of Australian students on international assessments of mathematics (e.g. Thompson, De Bortoli, Underwood, & Schmid, 2019) and the significant decline in the proportion of Year 12 students undertaking the more advanced mathematics courses. But the research also reveals significant inequalities in that students from low socioeconomic communities are far more likely to be represented in the 45 to 55% range of students not having access to multiplicative thinking than students from higher socioeconomic backgrounds, who are more likely to be represented in the 25 to 35% range. This situation is untenable where the fastest growing employment opportunities require some form of STEM qualification.

What can be done to support the development of multiplicative thinking?

Identifying and building on what students know in relation to important mathematics is widely regarded as the key to improving learning outcomes (e.g., Black & Wiliam, 1998; Goss, Hunter, Romanes & Parsonage, 2015; Masters, 2013; Timperley, 2009; Wiliam, 2011).

Moreover, where teachers are supported to identify and interpret student learning needs, they are more informed about where to start teaching, and better able to scaffold their students' mathematical learning (Callingham, 2010; Clarke, 2001; Siemon, 2016).

In response to the initial research project that identified multiplicative thinking as the source of the seven-year range in mathematics achievement (Siemon & Virgona, 2001), the *Scaffolding Numeracy in the Middle Years* (SNMY) project (2004-2006) used rich tasks and Rasch modelling to investigate the development of multiplicative thinking in just over 3200 students in Years 4 to 8 (Siemon & Breed, 2006; Siemon, Breed, Dole, Izard, & Virgona, 2006). The following resources were developed as a result of the project.

- **A Learning and Assessment Framework for Multiplicative Thinking (LAF)** that comprises an evidenced-based, eight-level learning progression for multiplicative thinking that describes a range of behaviours from additive, count all strategies (Zone 1) to the sophisticated use of proportional reasoning (Zone 8) with multiplicative thinking not evident on a consistent basis until Zone 4. Detailed targeted teaching advice that provides information on what needs to be consolidated and established at each Zone as well as what needs to be introduced and developed to scaffold student learning to the next Zone is also provided (see below)
- **Two validated assessment options** consisting of an extended task and five or six shorter tasks each of which contain two or more items. Partial credit scoring rubrics that value core knowledge, the ability to apply that knowledge, and the capacity to explain and justify are provided as well as two Raw Score Translators that map student scores to the one of the Zones of the learning progression
- **Additional Zone-based resources** were also provided in the form of learning plans and authentic tasks.

The SNMY project also demonstrated that teaching targeted to individual student learning needs can make a significant difference. For example, Breed (2011) undertook a doctoral study as part of the SNMY project. Nine Year 6 students identified in Zone 1 of the LAF in 2004 participated in an 18-week intervention in mid 2005. The students worked with the teacher in small groups using manipulatives, games, discussion and weekly written reflections using the LAF as a guide. When re-assessed three months after the intervention, all nine students shifted at least 4 zones with the majority shifting five Zones to be age and grade appropriate.

Targeted Teaching

Targeted teaching is a form of differentiation that is specifically concerned with students' learning needs in relation to a small number of **'big ideas'** in Number, in this case, multiplicative thinking, without which their progress in school mathematics will be seriously impacted (Siemon, 2006; 2017; Siemon, Bleckly, & Neal, 2012).

Targeted teaching is based on the premise that there are three key processes in involved in improving a student's mathematics learning:

- understanding where the learner is right now,
- understanding where the learner needs to be, and
- understanding how to get there (Wiliam, 2013)

Targeted teaching requires:

- **access to accurate information** about what each student is able to do (i.e., reliable, evidence-based eliciting tools)
- **interpretations of student behaviour** in terms of the key steps in the development of important mathematical ideas and strategies
- a **commitment to acting on the evidence** to inform both in-the-moment and future teaching (i.e., to use the evidence obtained to better target the learning needs of all students)
- an **expanded repertoire of teaching approaches** that accommodate and nurture discourse, help uncover and explore students' ideas in constructive ways, and ensure all students can participate in and contribute to the enterprise; and
- **flexibility** to spend time with those who need it most (Siemon, 2006)

The targeted teaching cycle for multiplicative thinking using the SNMY resources is shown in Figure 2 below.

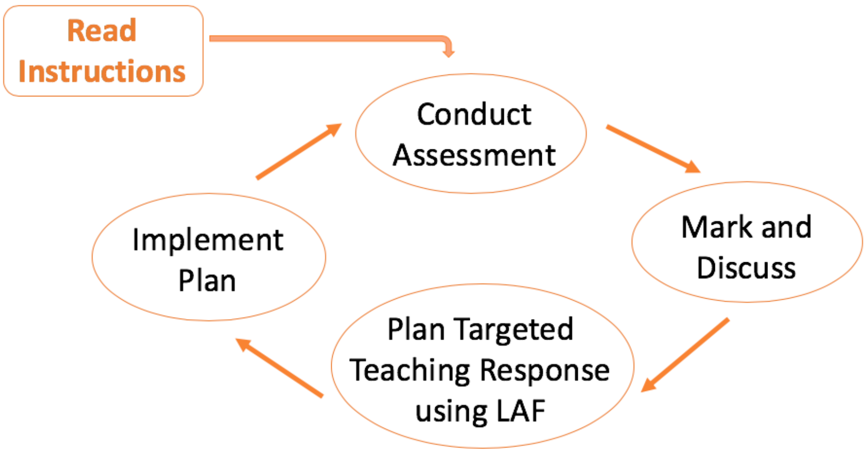


Figure 2. The targeted teaching cycle

Targeting Multiplicative Thinking Works

Targeted teaching is not easy but where implemented effectively, it can make a significant difference to student mathematics learning outcomes.

2006 – overall medium to large effect sizes[2] (in the range 0.45 to 0.75 or more) were found across the SNMY research schools (17 primary, 3 secondary) compared to small to medium effect sizes (in the range of 0.2 to 0.5) in the reference schools (Siemon, Breed, Dole, Izard, & Virgona, 2006).

2011 – Breed (2011) reported shifts of up to four Zones as a result of a targeted, 18-week intervention based on the Learning and Assessment Framework for Multiplicative Thinking.

2013 – the results of the *Reframing Mathematical Futures - Priority (RMF-P)* project demonstrated the efficacy of adopting a targeted teaching approach to multiplicative thinking using the SNMY materials in Years 7 to 9 (e.g., Siemon, 2016; Siemon, Banks, &

[2] An effect size of 0.4 or greater is considered to represent an improvement above what might otherwise be expected (Hattie, 2012).

Prassad, 2018). The average effect size across the 28 schools was 0.64, however, individual school results ranged from 0.4 to 1.2 (see Case Study 1 below).

2015 – the Grattan Institute report on *Targeted Teaching: How better use of data can improve student learning* (Goss, Hunter, Romanes, & Hunter, 2015) presents the general case for formative assessment and three case studies to showcase the benefits of adopting a targeted teaching approach

2016 – of the 10 schools that used the SNMY materials in Years 7 and 8 in the context of the *Reframing Mathematical Futures II* project (e.g., Siemon, 2019; Siemon, Banks, & Prassad, 2018), the average effect size was 0.47. Again, individual school results ranged considerably, but four schools achieved effect sizes well in excess of 1.0 (e.g. see Case Study 2 below).

A range of factors were nominated by the teachers involved in the RMF projects as reasons for the differential results. These included the extent to which the targeted teaching approach was endorsed and practically supported by school leadership, the availability of planning and professional learning time, access to appropriate spaces and resources, and the varying levels of staff 'buy in'. However, the teachers also reported that working together to moderate and discuss student responses was one of "the best professional development opportunities" they had experienced (Siemon, 2019).

Instructions for administering the Assessment Options

The purpose of the assessments is to find out what students know and can do, beyond whether they get the correct answers. Each task is marked using a detailed scoring rubric provided with the assessment options. The total score obtained by a student can be mapped to the *Learning Assessment Framework for Multiplicative Thinking* (LAF) using the Raw Score Translator for that option.

Please read the following instructions carefully before using any of the Assessment Options for Multiplicative Thinking.

Before using the Assessment Option

Allocate sufficient time - For the assessment to be a valid reflection of students' multiplicative thinking, it is essential that all students have sufficient time to do as much as they can on each task.

The tasks have been designed to be given over three to four sessions within a 1 to 2-week period. For instance, many teachers do the extended task in one teaching session then one or two of the shorter tasks at the start of subsequent teaching sessions.

While teachers may choose to do more than one task per session, it is suggested that no more than two tasks be attempted in any one session unless the session is more than one hour long. In general, 30 minutes seems to be sufficient for most students to do what they can on the extended task and 10-15 minutes seems to be sufficient for student to do as much as they can on the shorter tasks.

Prepare the materials - You will need to photocopy as many copies of the assessment tasks as needed. These should be prepared as booklets (i.e., printed and stapled) so that individual student work can be kept together [Note: students do not need copies of the Scoring Rubrics, Student Score sheet or Raw Score Translator).

Prepare the class - Treat this as you would a normal classroom activity. Try to avoid using the word 'test' and stress that the purpose of doing this is to inform future teaching.

Students should have access to pens, pencils, and erasers. Rulers may be used but they are not essential. Calculators and rulers are **not** needed.

Use the Sample Question - Many students are reluctant to write explanations or show their working and need to be encouraged to provide as much evidence of their mathematical thinking as possible.

The worked example below should be discussed with students to make sure that they understand what is expected of them prior to the assessment. Show and discuss the four student responses and use the scoring rubric with the class to score each response, noting that diagrams, words or symbols may be used.

In particular, it is important that students understand what is meant by the instructions:

- "Show all your working and explain your answer in as much detail as possible."
- "Explain your reasoning using as much mathematics as you can."
- "Use as much mathematics as you can to support your answer."

SAMPLE QUESTION

A gecko is about 8 cm long.

A frilled-neck lizard is about 6 times as long as a gecko.

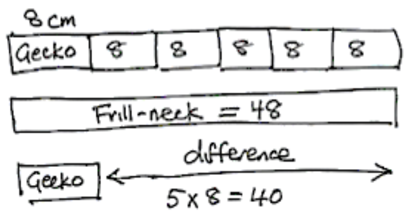
The difference between the length of a frilled-neck lizard and a gecko is about

2 cm	14 cm	40 cm	48 cm
○	○	○	○

Explain your reasoning using as much mathematics as you can (you may use a diagram if you wish)

(ACARA, 2013)

Four Student Responses:

Student 1.	Student 2.
40 cm	*40 cm because I added them and subtracted*
Student 3.	Student 4.
40 cm. Frill neck is 6 geckos so 6 X 8 = 48. Difference is 48 - 8 = 40	

Scoring Rubric:

0	No response or irrelevant response
1	Correct (40 cm) but no reasoning or explanation provided
2	Correct, incomplete reasoning or an operational description given
3	Correct, correct reasoning using words, diagram or symbols

Using the Assessment Option

Distribute the booklets. Stress that the purpose of doing this is to inform future teaching – it is in the student's best interests to do as well as they can and not copy. Go through the instructions on the second page of the assessment booklet.

Encourage working - Students are expected record all of their work in the Assessment Option booklet so there is **no need** for scrap paper or jotters etc. Encourage students to explain their reasoning using words, diagrams, or equations.

Additional space should not be needed but if so, advise students to use the back of the previous page (if booklets single sided) or find another space but make sure work is labelled. A single line should be placed through any rejected work (i.e. not obliterated or rubbed out) as it could provide some clues to students' thinking.

Student support - The object of the exercise is not that students get the right answer, but that they are given an opportunity to demonstrate what they actually do know and can do largely on their own.

Teachers can support students by answering questions without telling them what to do. Avoid providing so much support that students are able to complete the task with little understanding of what they are doing or why.

Teachers may:

- read the task to any student with reading problems
- scribe an oral explanation for students whose thinking may not otherwise be fairly represented
- explain unusual words as required.

Keep unfinished Option booklets in a safe place and ensure as far as possible that all students have an opportunity to attempt all tasks.do as much as they possibluy can

After using the Assessment Option

Collect booklets – Before collecting the booklets for the final time, make sure that each student has had sufficient time to do as much as they possibly can – this is about determining what they know and can do – it should not be what they can do in a set period of time.

Mark student work – wherever possible work with colleagues to do this using the option-specific Scoring Rubrics (included with each Assessment Option). Record student scores on the Student Score Sheet, noting any interesting responses/observations in the comments column.

Match to LAF - When the marking is completed, the student's total score can be compared to the option-specific Raw Score Translator (included with each Assessment Option). This will assign the student's performance to a Zone in the Learning Assessment Framework for Multiplicative Thinking.

Note: There may be a small number of students who receive a zero score or a perfect score. Assuming this represents the best they can do, all that can be said about these students is that they are either below Zone 1 or above Zone 8.

Where to next?

Refer to the teaching advice, that is, the **Learning Assessment Framework for Multiplicative Thinking 2021 (LAF)** to determine a starting point for teaching and/or targeted intervention. See below for a description and link to this and the related resources.

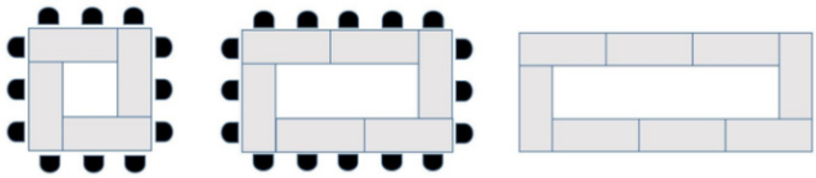
Student work samples

The following responses are provided as a guide to the use of the Scoring Rubrics.

Board Room Tables ...

The diagrams below show how these tables can be arranged for different numbers of people.

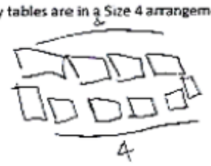
(No one sits inside the arrangement.)



Size 1	Size 2	Size 3
4 tables	6 tables	tables
12 people	16 people	people

[ABRT2]

How many tables are in a Size 4 arrangement? 10 tables

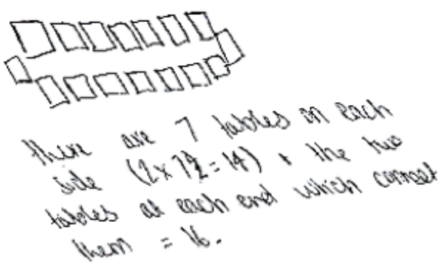


This item is scored on a 0,1 basis. The response is correct, so it is scored as a 1

[ABRT3]

How many tables are in a Size 7 arrangement? 16 tables

Explain your reasoning.



This item is scored on a 0,1,2 basis. The response is scored as a 2 as although a diagram is not mentioned in the rubric, it is supported by words

[A8RT4]

Write down in words or symbols a rule for working out the number of tables when you know the Size number.

Size number x 2 = ans
ans x2 = number of tables.

This item is scored on a 0,1,2 basis. The response is "correct, evidence of multiplicative thinking expressed in words (e.g., *two times the Size number plus two*) or in symbols (e.g., N = 2 x size + 2)" so it is scored as a 2.

[A8RT5]

Write down in words or symbols a rule for working out the table Size given the number of tables.

$\frac{\text{Number of tables}}{2}$ = ans

ans - 2 = table size

This item is scored on a 0,1,2 basis. It is scored as a 2 as there is evidence of multiplicative thinking expressed in words.
However, it is worth noting that this student appears to have difficulty constructing/expressing a pattern more formally

[A8RT6]

John said he needed 13 tables to set up for his meeting. Could John be correct? Explain how you know.

No, he could not as the amount of tables need for a size of 6 would be 14 and the tables needed for a size of 5 is 12, there is no inbetween.

This item is scored on a 0,1,2,3 basis. It is scored as a 3 as it recognises the relationship between table size and number of tables

[A8RT7]

What Size arrangement is needed to seat 72 people?

Explain your reasoning.

72 - 4 = 68

$\frac{68}{2}$ = 34

size 34 is needed.

This item is scored on a 0,1,2.3 basis. It is scored as a 1 as although it is incorrect, it is not entirely irrelevant.
This response is worth investigating, it appears as though the student has used an inappropriate generalisation.
Note also, the two-step solution rather than the use of a pattern

[A8RT8]

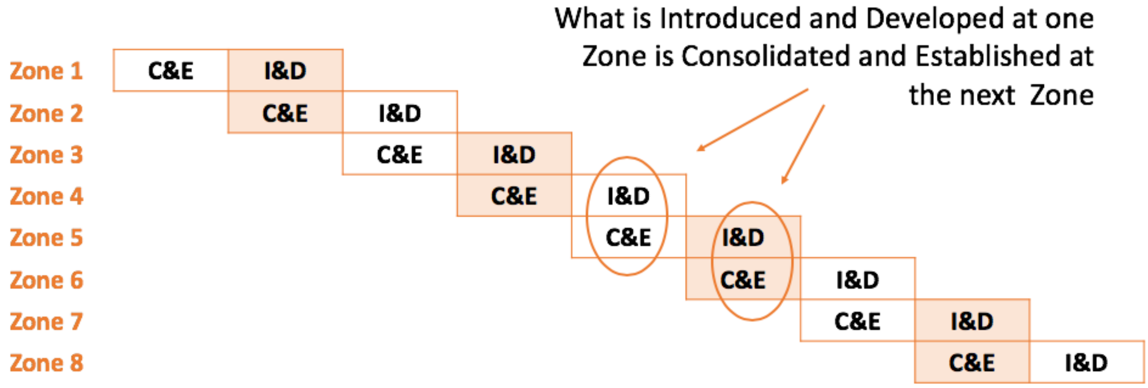
Write down in words or symbols a rule for working out the Size of the arrangement when you know the number of people.

number of people - 4 = ans

$\frac{\text{ans}}{2}$ = size of arrangement.

This item is scored on a 0,1,2.3 basis. It is scored as a 1 as although it is incorrect, it recognises that division and subtraction are involved.
This response is worth investigating for the same reasons as above

The Learning and Assessment Framework for Multiplicative Thinking (LAF)



The revised version of the LAF can be found here <<insert link>>. The LAF provides teaching advice to support a targeted teaching approach to multiplicative thinking. It should be used as the first 'port of call' in deciding how best to support student learning. As students are located at the point on the learning progression where they have a 50% chance of successfully completing the items at that level of difficulty, the advice for each Zone is presented in terms of what needs to be Consolidated and Established and what needs to be Introduced and Developed to scaffold students' progression to the next Zone.

It is important to note that the advice about what is introduced and developed at one Zone (e.g. Zone 4) is the same as the advice about what needs to be consolidated and established at the next Zone (e.g. Zone 5).

Support Material - the SNMY site contains a number of other resources from the original project, specifically, Learning Plans and Authentic Tasks. While these can be found at https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/resourcelibrary.aspx, they have been extensively reviewed both as a result of the RMFII project and more recently as part of the Growing Mathematically project. These resources have now been updated and linked to exemplary resources such as reSolve and maths300. They are now available on the AAMT Growing Mathematically website as **Zone -Based Targeted Teaching Activities** <<insert link here>>

Case Studies

The following case studies are taken from the Reframing Mathematical Futures projects, the Priority project in 2013 (RMF-P) and the subsequent RMF II project in 2015-2018. They are adapted from the ones reported in Siemon, Banks, & Prasad (2018).

Case Study 1 - Palberton Middle School (a pseudonym) is located in a growing, outer suburb of a northern Australian city. At the time, the school had 560 Year 7 to 9 students from a diverse range of cultural backgrounds. The school leadership team and the maths staff were keen to improve mathematics learning outcomes so 'jumped at the chance' to participate in the RMF-P project as they could see this working well with their commitment to team teaching and using data to inform teaching approaches. After attending the initial workshop in Melbourne in July 2013, the specialist teacher in consultation with the school leadership and two other maths staff decided to target four of the Year 8 classes (50% of the cohort) in what remained of the 2013 school year.

The school's purpose-built accommodation facilitated team teaching approaches. Four classrooms were grouped around a central covered space with large sliding doors providing access to the central space from each classroom. Co-teaching arrangements were formalised in 'hub' agreements and the team co-taught two of the four classes while a parallel team of English teachers co-taught the other two classes. Learning support staff were available on most occasions to support the work of the co-teaching teams. The school timetable provided five 50-minute lessons per week for maths (and English) which included one double lesson.

The RMF team as it became known at the school, administered, marked and moderated SNMY Option 1 for the four Year 8 classes in August 2013 and created profiles for all students. The specialist shared the data with the school leadership team and a key figure in the Department of Education, who were "shocked" to see that 53% of the Year 8s assessed were in Zones 1 to 3 of the LAF. When the leadership group recognised what this meant, further in-kind resources were made available to support the work of the project.

A decision was made to use the double period in maths each week to implement a targeted teaching approach to multiplicative thinking. These lessons, which came to be referred to as RMF Maths, were structured to include a Do Daily session, an open-ended problem related to the mathematics being considered in the other three lessons, work in Zone groups on targeted teaching activities, and a formal period of reflection. The approximate time spent on each of these components was 10, 40, 40 and 10 minutes respectively. Each member of the team was responsible for two to three Zones. The team met weekly to plan Zone activities, many of which they adapted to be age-appropriate and met again on Saturdays for professional sharing and forward planning. The students were given project books which they decorated in which to record their reflections at the end of each RMF lesson. A template was provided to help structure the reflections. The booklets were collected and reviewed by the team who used the reflections to inform their planning before being returned at the beginning the next lesson with written. The team observed considerable changes in the nature and amount of reflective comments provided by the students over the course of the semester – the students looked forward to reading the feedback from the teachers and quickly settled in order to see what was written.

The targeted teaching activities and materials were organised and stored by Zone in the hub area in open shelving that was available to students. This enabled some level of choice if students wanted to move on to another activity or try an activity from another Zone. While this was a massive effort, the teachers felt it was worth as one of the first things they noticed were that there were far fewer instances of challenging behaviour to deal with and students were asking if they could do 'RMF maths' all of the time. Another positive outcome was that students were becoming more metacognitive in their responses to problems they were doing in the non-RMF lessons, for instance, the team noticed that many of the students started to explain their reasoning without being asked. Although the demands on the teaching staff were high with many additional hours per week spent on preparing and adapting Zone activities, the teachers felt that they had grown as a team and were more knowledgeable about how to deal with student misconceptions.

SNMY Option 2 was administered in November and marked and moderated by the co-teaching team. The data were de-identified, recorded on a spreadsheet and forwarded to the research team for analysis. Data from 70 matched pairs were available for analysis, the results of which can be seen in Figure 3. The improvement in multiplicative thinking was impressive with an adjusted effect size of 1.18.

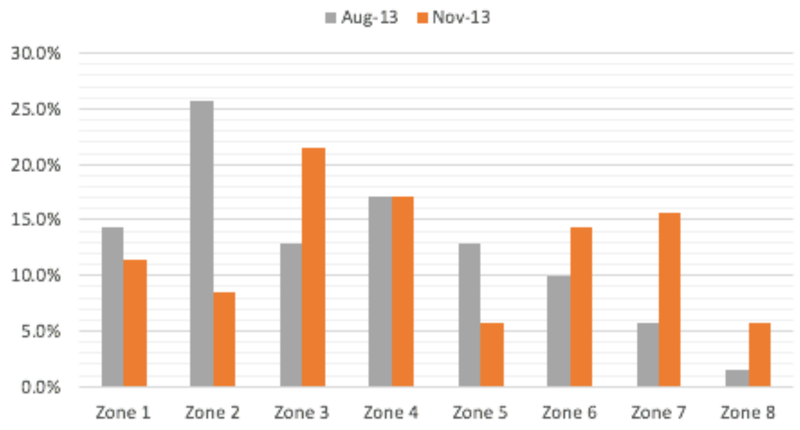


Figure 3. Proportion of Year 8 students by LAF Zone in August and November (n = 70)

Case Study 2

Plumpton High School is located in an established western suburb of Sydney. It is a large multi-cultural 7 to 12 secondary school, a key goal of which is to "put students first". Plumpton High School came to use the SNMY materials and implement a targeted teaching approach to multiplicative thinking as a result of the school's participation in RMFII which was aimed at building a similar evidenced-based framework for mathematical reasoning in Years 7 to 10. As one of the 'new' schools that were unfamiliar with the notion of a targeted teaching, the school was asked to use the SNMY materials to identify and respond to student learning needs in relation to multiplicative thinking before contributing to the trial of the mathematical reasoning materials.

When offered the opportunity to participate in RMFII project in late 2014, the mathematics results at the school was a concern and number of students electing to pursue the more advanced maths courses in the senior years was declining. The school felt a change was needed. As a result, the school leadership not only agreed to participate in the project they decided to send an additional teacher to the initial three-day workshop in Melbourne in November 2014 at the school's expense that introduced teachers from the 'new' schools to multiplicative thinking and the SNMY materials. The RMF-P specialists who were continuing in the follow up project were able to share their SNMY results and describe what worked and what did not work in implementing a targeted teaching approach to multiplicative thinking in secondary school contexts. Key strategies that were variously adopted by the RMF-P schools that were most successful included team teaching, dedicated lesson times for targeted teaching and Zone-based activities, locally available resources, team planning time, additional time release, access to professional learning opportunities, and support of school leadership. RMF-P schools implemented these and other strategies to different extents and in different ways appropriate to their circumstances but the teachers from the 'new' schools such as Plumpton were able to draw on this information to plan how they would implement a targeted teaching approach.

On returning to school, a decision was made to focus on the whole of Year 8 in 2015. Teaching staff felt that the current Year 7 students would most benefit from the intervention and as they were still at school it would make sense to administer SNMY Option 1 in December of 2014. The school leadership supported the decision to focus on Year 8 in 2015 as this cohort would sit the NAPLAN test in Year 9 in 2016 which would provide an independent evaluation of the intervention.

In 2015, each of the six Year 8 classes had a separate 75-minute RMF lesson per week. During this time, the students worked in their Zone groupings initially on activities from the project Dropbox and/or ones prepared by the specialist. The specialist and one other of the senior maths teachers, dropped by the classrooms whenever they could to help and prepared resources in their free periods. As time went on and the demand for new, age-appropriate activities increased, the Year 8 teachers also developed and shared Zone-based activities with their colleagues. One of the ways in which this happened was at the Wednesday lunches, where Year 8 staff talked about what they were doing, reflected on progress and developed new ideas. A lesson template was developed and staff would workshop new lessons prior to delivery. Referred to as 'Live in Lessons', this enabled the team to iron out any potential issues and to make links to regular classroom teaching activities and content.

While project funding was provided to support the implementation of a targeted teaching approach to multiplicative thinking in four classes, the school decided to implement this approach in 6 classes of 30 students, which meant resources were tight. Priority was given to purchasing concrete materials and a separate area was set up to keep class booklets, resources and activities for easy collection and distribution. Initially, there was little buy-in from students and teachers as working in groups was something new for many. The existing class structure (semi-streamed) helped manage the targeted teaching approach but there was considerable variation in each classroom. Planning was essential and proved to be a key factor to the school's success. Over the course of the year, teachers found that they were incorporating many of Zone type activities into the curriculum being taught in the week, placing particular emphasis on the need to explain and justify solution strategies as this had proved to be a major sticking point early on. The team learnt as they went and kept on sharing, adjusting and implementing strategies/activities which worked in other classes. Staff meetings on Mondays were focussed on developing teachers' capacity to share resources and ideas to help the growth of targeted teaching in classrooms.

Gradually, everything became easier, the students were more accustomed to working in groups and appreciated the opportunity to experience success. Student engagement increased and the quality of their responses to school-based assessments improved noticeably. Teaching staff were more inclined to design reasoning activities for regular classroom teaching and provide time for students to apply what they know in unfamiliar contexts and marking rubrics were slowly incorporated into classroom assessment tasks.

SNMY Option 2 was administered, marked and moderated by Year 8 teachers and the two specialists in September 2015. The results were again de-identified and forwarded to the research team for analysis. The results were impressive and immediately bought buy in from senior management and other maths teachers. Additional teacher release was provided to support the preparation of resources, marking and moderating of assessments, and training of other staff members. The growth is shown in Figure 4 and represents an effect size of over 3.5.

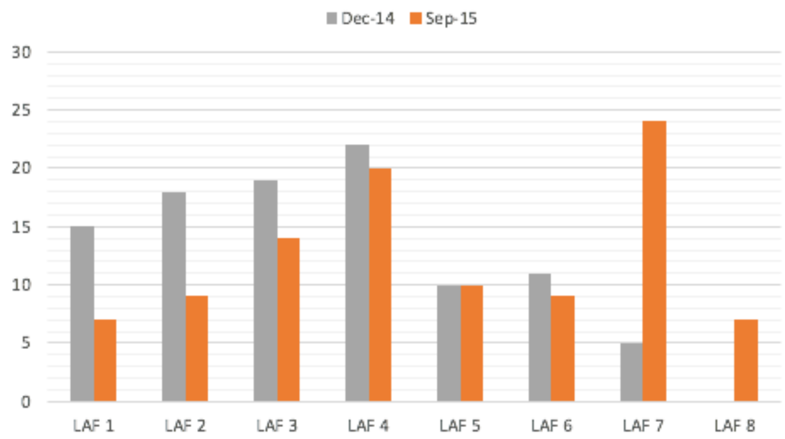


Figure 4. Proportion of Year 8 students by LAF Zone (141<n<152)

While not the only measure of success in school mathematics, the Year 9 NAPLAN results for the same cohort in 2016 provide conclusive evidence that targeted teaching makes a difference. Compared to the previous Year 9 who sat the NAPLAN test in 2015, the average scaled growth score for the school went from below all State in 2015 (45.6) to above all State in 2016 (51.1). But perhaps more telling are the respective growth comparisons between 2015 and 2016 of the proportion of students in the less than expected growth category versus the proportion of students in the greater than or equal to expected growth category.

Table 2. Growth comparisons for the 2015 and 2016 Year 9 cohorts, Plumpton High School

	Less than Expected growth	Greater than or equal to expected growth
2015	48.7%	51.2%
2016	34.5%	65.5%

Case Study 3

Deloraine High School is a rural Year 7-12 Secondary School located 50km out of Launceston, Tasmania. The school implemented the Reframing Mathematical Futures II resources in Years 7, 8 and 9 with each grade cohort consisting of approximately 45 – 60 students in two classes per grade. In each grade, students undertake three 105 minute sessions of Mathematics per week and to implement the program one lesson per week was allocated. Over the four terms, students in grade 7 and 8 focused on multiplicative thinking for approximately 18 weeks and algebraic reasoning, statistical reasoning and geometric reasoning for 8 weeks each, whilst students in grade 9 engaged with resources focusing on algebraic, statistical and geometric reasoning for 10 weeks each. The resources were coupled with 'normal' mathematics lessons over the course of the year where students engaged with direct teaching, guided instruction and individual and collaborative projects mapped against the Australian curriculum requirements for each grade.

At the beginning of Grade 7 students undertook the Scaffolding Numeracy in the Middle Years – Assessment Option 1 and at the end of the multiplicative thinking program (18 lessons) students were assessed against the Assessment Option 2 to monitor and record progress. Students then completed Assessment Option 1 again at the conclusion of the multiplicative thinking program (18 lessons) in Grade 8, to further monitor and record progress. This enabled students to be mapped against the Learning and Assessment Framework (LAF) on three separate occasions, which provided suggestions for targeted teaching activities aimed at improving each students' ability to think multiplicatively. This enabled the school to group students in Grades 7 and 8 for the targeted teaching lessons into 5 distinct groups per grade. Significantly, support from the school leadership team enabled each group to be led by a trained teacher including the Principal herself, the Assistant Principal – Numeracy and the Advanced Skills Teacher (Head of Mathematics Department). This in itself made visible to students that the school valued the importance of the program and therefore encouraged their own best effort when engaging with the resources. Each of the targeted groups (Grade 7 and Grade 8) engaged in differentiated content against key themes in the multiplicative thinking unit, namely (two lessons on factors, partitioning, decimals and fractions, place value, ratio, proportion, Cartesian product and generalisation of pattern). The resources involved activities drawn directly from the teaching and learning advice in the LAF and was supplemented by resources from well researched and validated organisations including Maths 300, the reSolve: maths by inquiry project (AAMT), the nRich Maths Project (Cambridge University) and NCTM illuminations. Similarly, as Grade 7, 8 and 9 students engaged in algebraic, statistical and geometric reasoning targeted teaching lessons, students engaged in pre and post-assessment tasks using the Assessment options produced in the RMF II project and resources were developed and implemented according to student ability. Whilst this is still evolving each year, the benefits to students have been significant and teachers have also indicated greater confidence in their own practice and pedagogical content knowledge.

References

Black, P., & Wiliam, D. (1998). Inside the black box: Raising standards through classroom assessment. *Phi Delta Kappa*, *80*(2), 139–147.

Breed, M. (2011). *Constructing paths to multiplicative thinking* [Unpublished PhD Thesis]. RMIT University.

Callingham, R. (2010). Mathematics assessment in primary classrooms. In ACER (Ed.), *Proceedings of 2010 ACER Research Conference* (pp. 39-42), Melbourne: ACER.

Goss, P., Hunter, J., Romanes, D., & Parsonage, H. (2015). *Targeted teaching: how better use of data can improve student learning*. Grattan Institute.

Lamon, S. (2012). *Teaching fractions and ratios for understanding—Essential content knowledge and instructional strategies for teachers*. (3rd ed.). Lawrence Erlbaum Associates.

Masters, G. (2013). Reforming educational assessment: Imperatives, principles and challenges. *Australian Education Review, No. 57*, 1-57.

Siemon D. (2016). Addressing the STEM Challenge though targeted teaching – What's the evidence? In *Proceedings of the 2016 Research Conference: Improving STEM learning- What will it take?* Melbourne: ACER.

Siemon, D. (2013). Launching Mathematical Futures: The key role of multiplicative thinking. In S. Hernert, J. Tillyer & T. Spencer. (Eds.), In *Mathematics: Launching futures, Proceedings of the 24th Biennial Conference of the Australian Association of Mathematics Teachers*, (pp. 36-52). Adelaide: AAMT.

Siemon, D. (2017) Targeting the big ideas in mathematics. Teacher Evidence + Insight+ Action, ACER (Feb 27), https://www.teachermagazine.com.au/articles/targeting-big-ideas-in-mathematics

Siemon, D. (2019). Knowing and building on what students know: The case of multiplicative thinking. In D. Siemon, T. Barkatsas & R. Seah (Eds.), *Researching and using progressions (trajectories) in mathematics education* (pp. 6-31). Brill-Sense.

Siemon, D. & Breed, M. (2006, November 26-30). *Assessing multiplicative thinking using rich assessment tasks* [Conference paper]. Annual Conference of the Australian Association for Research in Education, Adelaide. https://www.aare.edu.au/data/publications/2006/sie06375.pdf

Siemon, D. & Virgona, J. (2001). Roadmaps to numeracy – Reflections on the Middle Years Numeracy Research Project. *Proceedings of the Annual Conference of the Australian Association for Research in Education*, Fremantle. (www.aare.edu.au)

Siemon, D. & Virgona, J. (2002). Reflections of the Middle Years Numeracy Research Project – Is it a case of too much too soon for too many? In B. Barton, K. C. Irwin, M. Pfannkuch & M. O. J. Thomas (Eds.), *Mathematics education in the South Pacific - Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia* (pp.617-624). Auckland: MERGA.

Siemon, D., Banks, N., & Prasad, S. (2018). Multiplicative thinking a STEM foundation. In T. Barkatsas, N. Carr & G. Cooper (Eds.), *STEM education: An emerging field of inquiry*. Sense Publications.

Siemon, D., Bleckly, J. & Neal, D. (2012). Working with the big ideas in number and the Australian Curriculum Mathematics. In W. Atweh, M. Goos, R Jorgensen & D. Siemon (Eds.) *Engaging the Australian Curriculum Mathematics – Perspectives from the field* [Online Book], (pp. 19-46). Mathematical Education Research Group of Australasia. https://www2.merga.net.au/sites/default/files/editor/books/1/Chapter%202%20Siemon.pdf

Siemon, D., Breed, M., Dole, S., Izard, J., & Virgona J. (2006). *Scaffolding Numeracy in the Middle Years – Project findings, materials and resources, Final report*. Retrieved from https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/scaffoldnum.aspx

Siemon, D., Callingham, R., Day, L., Horne, M., Seah, R., Stephens, M., & Watson, J. (2018). From research to practice: The case of mathematical reasoning. In J. Hunter, L. Darragh & P. Perger (Eds.), *Making waves, opening spaces, Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 40-49). Auckland, NZ: MERGA.

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Mathematics in the Centre, Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 5, pp. 113-120), Prague: PME.

Thompson, S., De Bortoli, L., Underwood, C., & Schmidt, M. (2019). PISA 2018: Reporting Australia's results. Volume 1: Student Performance. https://research.acer.edu.au/ozpisa/35/

Timperley, H. (2009). Using assessment data for improving teacher practice. In ACER (Ed.), *Assessment and student learning: Collecting, interpreting and using data to inform teaching, Proceedings of 2009 ACER Research Conference* (pp. 21-25), Perth: ACER.

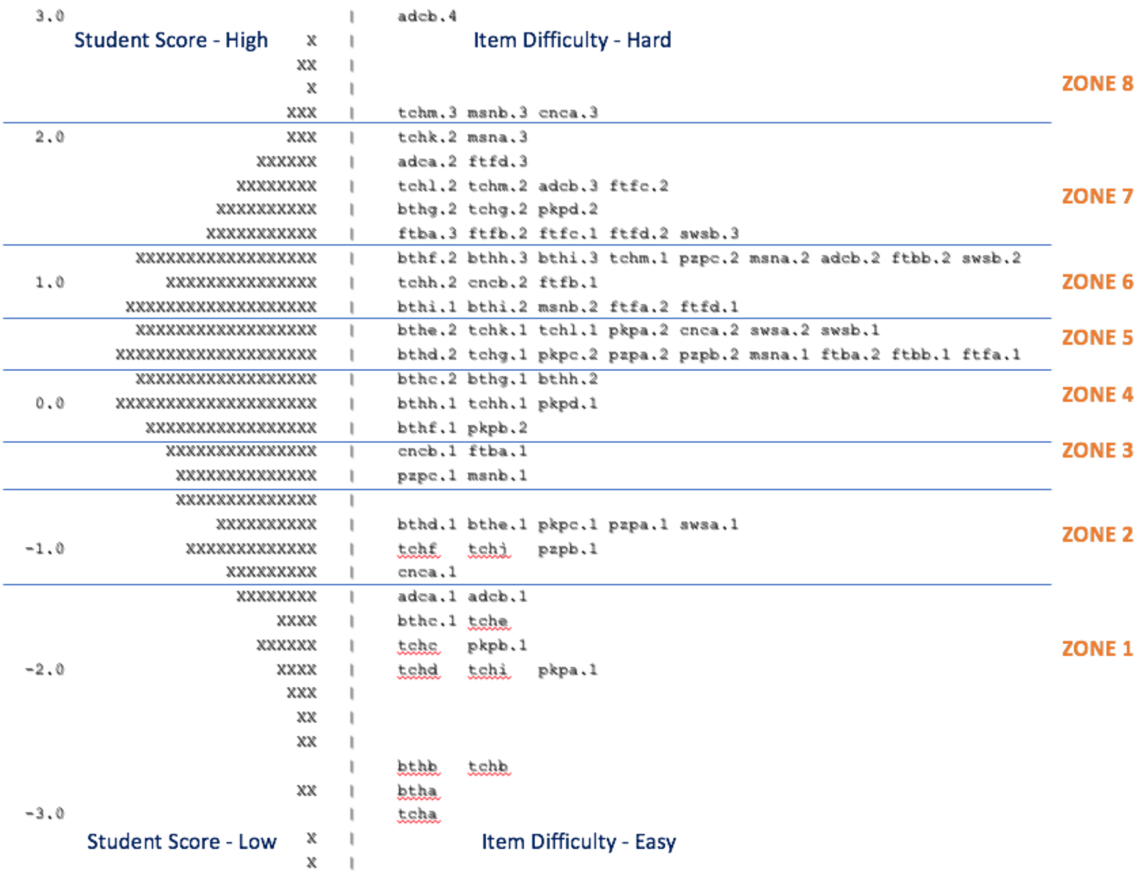
Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds), *Acquisition of mathematics concepts and processes* (pp. 127–173). Academic Press.

Wiliam, D. (2011). *Embedded formative assessment*. Solution Tree Press.

Wiliam, D. (2013). Assessment: The bridge between teaching and learning. *Voices from the Middle, 21*(2), 15 – 20.

Appendix 1

The analysis of the SNMY student data produced a map that related student scores (low to high) to item difficulties (easy to hard) as shown below. The item analysis facilitated the identification of an eight-level learning progression for multiplicative thinking that described a range of behaviours from additive, count all strategies (Zone 1) to the sophisticated use of proportional reasoning (Zone 8) with multiplicative thinking not evident on a consistent basis until Zone 4. It also supported the development of Zone-based teaching advice referred to as the *Learning and Assessment Framework for Multiplicative thinking* or LAF for short.



Variable Map SNMY Project 2006

As individual students are located on the same scale at the point where they have a 50% chance of successfully completing the items at that level of difficulty, the advice for each Zone is presented in terms of what needs to be consolidated and established and what needs to be introduced and developed to scaffold students' progression to the next Zone.