GROWING Mathematically

Multiplicative Thinking

Teacher's Manual

(Trial Version)

Growing Mathematically is a project funded by the Australian Federal Government. It builds on materials that were developed as part of the RMIT Reframing Mathematical Futures II and the Scaffolding Numeracy in the Middle Years (SNMY) projects. SNMY content is currently hosted by the Victorian Department of Education.

GROWING MATHEMATICALLY: Multiplicative Thinking

(DRAFT) TEACHER MANUAL

Supporting a targeted teaching approach to multiplicative thinking in the middle years based on an evidenced-based learning progression

This resource has been produced by the Australian Association of Mathematics Teachers (AAMT), in collaboration with Emerita Professor Di Siemon of RMIT and her colleagues with funding from the Commonwealth Government of Australia.

The aim of the Manual is to add value to the existing formative assessment materials for multiplicative thinking developed by the *Scaffolding Numeracy in the Middle Years Project* (Siemon, Breed, Dole, Izard, & Virgona, 2006)[1].

The update has been made possible by the results of the *Reframing Mathematical Futures* (2013–2018) projects that explored the efficacy of using the SNMY materials in secondary schools alongside the development of similar formative assessment materials for algebraic, geometrical and statistical reasoning (Siemon, Callingham, Day, Horne, Seah, Stevens, & Watson, 2018).

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- **Why is MT important?** Brief description of the SNMY and RMFII projects, data to show MT the issue in the middle years and targeted teaching works!
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- **Targeted teaching –** description and evidence to show that TT works
- **Instructions –** how to administer and mark assessment options
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- **Learning Assessment Framework for Multiplicative Thinking** - learning progression, teaching advice, and link to related resources
- **Case Studies** – examples to show how schools have implemented a TT approach
- **ACARA Mapping –** alignment between MT Learning Progression, the *Australian Curriculum: Mathematics*, and the National Numeracy Continuum

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[1] https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/scaffoldnum.aspx

- **References/Further Reading** – links to relevant research papers, wider reading (to be completed)
- **Research Basis** – Appendices (to be added)

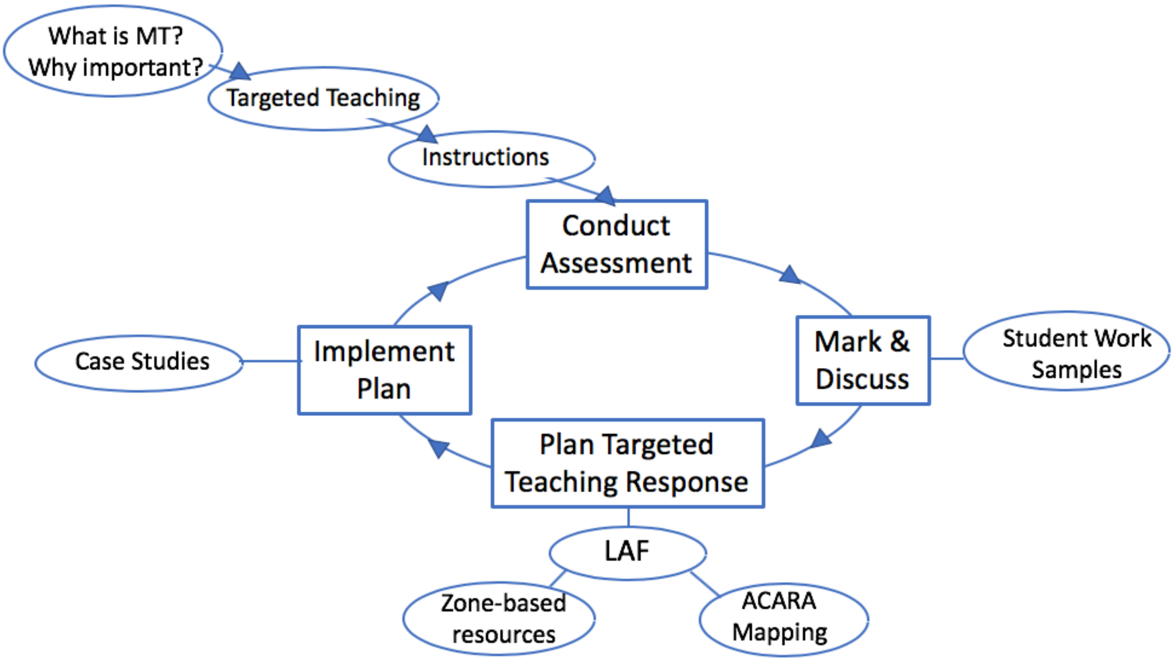

Figure 1. A conceptual map of the Growing Mathematically – Multiplicative Thinking Resource.

What is multiplicative thinking?

Multiplicative thinking involves recognising and working with relationships between quantities. Although some aspects of multiplicative thinking are available to young children, multiplicative thinking is substantially more complex than additive thinking and may take many years to achieve (Lamon, 2012; Vergnaud, 1983). This is because multiplicative thinking is concerned with processes such as replicating, shrinking, enlarging, and exponentiating that are fundamentally more complex than the more obvious processes of aggregation and disaggregation associated with additive thinking and the use of whole numbers (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015).

Multiplicative thinking is qualitatively different to additive thinking. It is evident when students:

- work flexibly and confidently with an extended range of numbers (i.e. larger whole numbers, fractions decimals, per cent, and ratios);
- solve problems involving multiplication and division, including direct and indirect proportion using strategies appropriate to the task; and
- explain and communicate their reasoning in a variety of ways (e.g. words, diagrams, symbolic expressions, and written algorithms. (Siemon, Breed, & Virgona, 2005).

In short, where additive thinking involves the aggregation or disaggregation of collections (e.g., $634 + $478 or finding the difference between 82 kg and 67 kg), multiplicative thinking involves reasoning with relationships between quantities, for example,

- 3 bags of wool per sheep, 5 sheep, how many bags of wool?,
- At an average speed of 85 km/hour, how long will it take to travel 367 km?.

Additive problems generally involve one measure space (e.g., dollars or kilograms) while multiplicative problems generally involve working with two (or more) measure spaces (e.g. bags of wool, number of sheep) and a relationship between the two (i.e. 3 bags of wool per sheep).

Because simple multiplicative problems such as '24 strawberry plants per row, 17 rows, how many strawberry plants?' can be solved additively using repeated addition or by using a learnt algorithm and known facts, it can be difficult to determine whether or not a student is thinking multiplicatively. Where this becomes apparent is where the problems involve larger whole numbers, fractions, decimals, per cent or ratios, and/or more complex relationships between quantities. For example, the following problems will generally provoke a range of strategies, not all of which are multiplicative

> A muffin recipe uses 2/3 cup of milk to make 12 muffins. How many muffins can be made with 6 cups of milk?
>
> A small business owner wants to offer a further 20% discount on her summer clothing range, but she needs to ensure she covers the wholesale price. The wholesale price of a summer top was $73. If the original price of the top was $139 and it was currently on sale for 30%, can she offer a 20% discount on the already discounted price?
>
> Mobile phone covers are offered in 5 different sizes, 3 different styles, and 14 different colours. How many different phone covers need to be ordered to have 3 of each type in stock?

> Sam said that doubling the dimensions of the garden box would double the volume. Is he correct? Use as much mathematics as you can to justify your conclusion.
>
> If it takes 3 men 24 hours to paint a house, how long will it take 2 men to paint the house?
>
> A wildlife officer estimated that there were 73 koalas in one forest reserve of 328 hectares and 62 in another forest reserve of 263 hectares. Which forest reserve provided more space for each koala?

Why is multiplicative thinking important?

Multiplicative thinking is crucial to success in school mathematics. It underpins nearly all of the topics considered in the middle years and beyond (see Siemon, 2013) and it is fundamental to careers in science, technology, engineering and mathematics (STEM).

Multiplicative thinking is needed to support efficient solutions to more difficult problems involving multiplication and division, fractions, decimal fractions, ratio, rates and percentage, and to solve proportional reasoning problems as they arise in algebra, geometry, measurement, statistics, and probability.

However, Australian research suggests that at least 25% and up to 55% of students in Year 8 do not have access to this critical capacity (Siemon, Breed, Dole, Izard, & Virgona, 2006; Siemon, 2013, 2016, 2019; Siemon, Banks, & Prasad, 2018).

A large-scale study involving just under 7000 Victorian students in Years 5 to 9 found that there was a seven-year range in student mathematics achievement in each year level, which was almost entirely due to the extent to which students had access to multiplicative thinking (Siemon & Virgona, 2001). More recent studies involving up to 32 secondary schools across Australia have confirmed that access to multiplicative thinking remains the reason for the significant difference in student mathematics achievement in Years 7 to 9 (e.g., Siemon, 2013, 2016, 2019; Siemon, Banks, & Prassad, 2018).

Lack of access to multiplicative thinking helps explain the reported decline in the performance of Australian students on international assessments of mathematics (e.g. Thompson, De Bortoli, Underwood, & Schmid, 2019) and the significant decline in the proportion of Year 12 students undertaking the more advanced mathematics courses. But the research also reveals significant inequalities in that students from low socioeconomic communities are far more likely to be represented in the 45 to 55% range of students not having access to multiplicative thinking than students from higher socioeconomic backgrounds, who are more likely to be represented in the 25 to 35% range. This situation is untenable where the fastest growing employment opportunities require some form of STEM qualification.

What can be done to support the development of multiplicative thinking?

Identifying and building on what students know in relation to important mathematics is widely regarded as the key to improving learning outcomes (e.g., Black & Wiliam, 1998; Goss, Hunter, Romanes & Parsonage, 2015; Masters, 2013; Timperley, 2009; Wiliam, 2011).

Moreover, where teachers are supported to identify and interpret student learning needs, they are more informed about where to start teaching, and better able to scaffold their students' mathematical learning (Callingham, 2010; Clarke, 2001; Siemon, 2016).

In response to the initial research project that identified multiplicative thinking as the source of the seven-year range in mathematics achievement (Siemon & Virgona, 2001), the *Scaffolding Numeracy in the Middle Years* (SNMY) project (2004-2006) used rich tasks and Rasch modelling to investigate the development of multiplicative thinking in just over 3200 students in Years 4 to 8 (Siemon & Breed, 2006; Siemon, Breed, Dole, Izard, & Virgona, 2006). The following resources were developed as a result of the project.

- **A Learning and Assessment Framework for Multiplicative Thinking (LAF)** that comprises an evidenced-based, eight-level learning progression for multiplicative thinking that describes a range of behaviours from additive, count all strategies (Zone 1) to the sophisticated use of proportional reasoning (Zone 8) with multiplicative thinking not evident on a consistent basis until Zone 4. Detailed targeted teaching advice that provides information on what needs to be consolidated and established at each Zone as well as what needs to be introduced and developed to scaffold student learning to the next Zone is also provided (see below)
- **Two validated assessment options** consisting of an extended task and five or six shorter tasks each of which contain two or more items. Partial credit scoring rubrics that value core knowledge, the ability to apply that knowledge, and the capacity to explain and justify are provided as well as two Raw Score Translators that map student scores to the one of the Zones of the learning progression
- **Additional Zone-based resources** were also provided in the form of learning plans and authentic tasks.

The SNMY project also demonstrated that teaching targeted to individual student learning needs can make a significant difference. For example, Breed (2011) undertook a doctoral study as part of the SNMY project. Nine Year 6 students identified in Zone 1 of the LAF in 2004 participated in an 18-week intervention in mid 2005. The students worked with the teacher in small groups using manipulatives, games, discussion and weekly written reflections using the LAF as a guide. When re-assessed three months after the intervention, all nine students shifted at least 4 zones with the majority shifting five Zones to be age and grade appropriate.

Targeted teaching

Targeted teaching is a form of differentiation that is specifically concerned with students' learning needs in relation to a small number of **'big ideas'** in Number, in this case, multiplicative thinking, without which their progress in school mathematics will be seriously impacted (Siemon, 2006; 2017; Siemon, Bleckly, & Neal, 2012).

Targeted teaching is based on the premise that there are three key processes in involved in improving a student's mathematics learning:

- understanding where the learner is right now,
- understanding where the learner needs to be, and
- understanding how to get there (Wiliam, 2013)

Targeted teaching requires:

- **access to accurate information** about what each student is able to do (i.e., reliable, evidence-based eliciting tools)
- **interpretations of student behaviour** in terms of the key steps in the development of important mathematical ideas and strategies
- a **commitment to acting on the evidence** to inform both in-the-moment and future teaching (i.e., to use the evidence obtained to better target the learning needs of all students)
- an **expanded repertoire of teaching approaches** that accommodate and nurture discourse, help uncover and explore students' ideas in constructive ways, and ensure all students can participate in and contribute to the enterprise; and
- **flexibility** to spend time with those who need it most (Siemon, 2006)

The targeted teaching cycle for multiplicative thinking using the SNMY resources is shown in Figure 2 below.

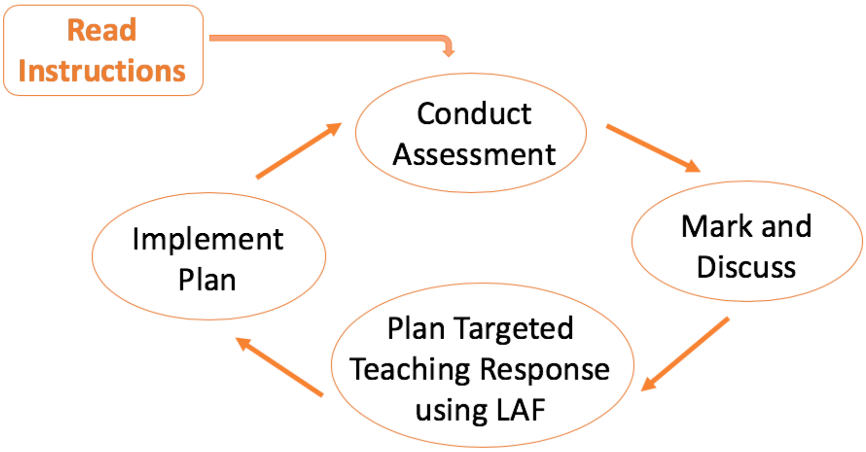

Figure 2. The targeted teaching cycle.

Targeting multiplicative thinking works

Targeted teaching is not easy but where implemented effectively, it can make a significant difference to student mathematics learning outcomes.

2006 – Overall medium to large effect sizes[2] (in the range 0.45 to 0.75 or more) were found across the SNMY research schools (17 primary, 3 secondary) compared to small to medium effect sizes (in the range of 0.2 to 0.5) in the reference schools (Siemon, Breed, Dole, Izard, & Virgona, 2006).

2011 – Breed (2011) reported shifts of up to four Zones as a result of a targeted, 18-week intervention based on the Learning and Assessment Framework for Multiplicative Thinking.

2013 – The results of the *Reframing Mathematical Futures - Priority (RMF-P)* project demonstrated the efficacy of adopting a targeted teaching approach to multiplicative thinking using the SNMY materials in Years 7 to 9 (e.g., Siemon, 2016; Siemon, Banks, & Prassad, 2018).

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[2] An effect size of 0.4 or greater is considered to represent an improvement above what might otherwise be expected (Hattie, 2012).

The average effect size across the 28 schools was 0.64, however, individual school results ranged from 0.4 to 1.2 (see Case Study, 1. p. 22).

2015 –The Grattan Institute report on *Targeted Teaching: How better use of data can improve student learning* (Goss, Hunter, Romanes, & Hunter, 2015) presents the general case for formative assessment and three case studies to showcase the benefits of adopting a targeted teaching approach

2016 – Of the 10 schools that used the SNMY materials in Years 7 and 8 in the context of the *Reframing Mathematical Futures II* project (e.g., Siemon, 2019; Siemon, Banks, & Prassad, 2018), the average effect size was 0.47. Again, individual school results ranged considerably, but four schools achieved effect sizes well in excess of 1.0 (e.g. see Case Study 2, p. 23).

A range of factors were nominated by the teachers involved in the RMF projects as reasons for the differential results. These included the extent to which the targeted teaching approach was endorsed and practically supported by school leadership, the availability of planning and professional learning time, access to appropriate spaces and resources, and the varying levels of staff 'buy in'. However, the teachers also reported that working together to moderate and discuss student responses was one of "the best professional development opportunities" they had experienced (Siemon, 2019).

Instructions for administering the Assessment Options

The purpose of the assessments is to find out what students know and can do, beyond whether they get the correct answers. Each task is marked using a detailed scoring rubric provided with the assessment options. The total score obtained by a student can be mapped to the *Learning Assessment Framework for Multiplicative Thinking* (LAF) using the Raw Score Translator for that option.

> Because this is a trial, students are asked to create a unique identifier that will be used to link pre- and post-intervention assessments. Students will be recognised by the project team through this identifier to ensure their privacy, hence the provision of an Excel spreadsheet. It is important that the spreadsheet only contains the unique student identifier and the student's score on each item.
>
> While trial school teachers are advised to keep the Student Record sheets for their own purposes – we ask that you send us photocopies of any unusual or interesting student responses. Please ensure that the student code is included on any work samples. These can be sent electronically with the spreadsheet or posted to the AAMT office.
>
> Please note that because this is a trial of new materials, it is possible that the Raw Score Translator may be slightly inaccurate. If there are groups of students who appear to be working way above or below your expectations, based on what you know about these students, please make a note of this and let the project team know. Your insights are important in refining the materials for other teachers to use. [*]

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[*] Text particular to the trial version of this document is styled in grey.

Please read the following instructions carefully before using the updated SNMY/RMFII Assessment Options for Multiplicative Thinking.

Before using the Assessment Option

Allocate sufficient time

For the assessment to be a valid reflection of students' multiplicative thinking, it is essential that they have sufficient time to do as much as they can on each task. The tasks have been designed to be given over three to four sessions within a 1 to 2-week period. For instance, many teachers do the extended task in one teaching session then one or two of the shorter tasks at the start of subsequent teaching sessions.

While teachers may choose to do more than one task per session, it is suggested that no more than two tasks be attempted in any one session unless the session is more than one hour long. In general, 30 minutes seems to be sufficient for most students to do what they can on the extended task and 10-15 minutes seems to be sufficient for student to do as much as they can on the shorter tasks.

Prepare the materials

For the purposes of the trial, you will be provided with either Option 3 or Option 4.

You will need to photocopy as many copies of the assessment tasks as needed including the two blank pages at the end. These should be prepared as booklets (i.e., printed and stapled) so that individual student work can be kept together (Note: students do not need copies of the Scoring Rubrics, Student Score sheet or Raw Score Translator).

Prepare the class – Treat this as you would a normal classroom activity. Try to avoid using the word 'test' and stress that the purpose of doing this is to inform future teaching.

Students should have access to pens, pencils, and erasers. Rulers may be used but they are not essential. Calculators and rulers are **not** needed.

Use the Sample Question – Many students are reluctant to write explanations or show their working and need to be encouraged to provide as much evidence of their mathematical thinking as possible.

The worked example below should be discussed with students to make sure that they understand what is expected of them prior to the assessment. Show and discuss the four student responses and use the scoring rubric with the class to score each response, noting that diagrams, words or symbols may be used.

In particular, it is important that students understand what is meant by the instructions:

- "Show all your working and explain your answer in as much detail as possible."
- "Explain your reasoning using as much mathematics as you can."
- "Use as much mathematics as you can to support your answer."

SAMPLE QUESTION

A gecko is about 8 cm long.

A frilled-neck lizard is about 6 times as long as a gecko.

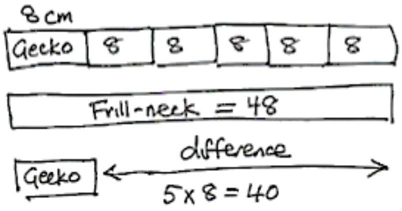
The difference between the length of a frilled-neck lizard and a gecko is about

2 cm	14 cm	40 cm	48 cm
○	○	○	○

Explain your reasoning using as much mathematics as you can (you may use a diagram if you wish)

(ACARA, 2013)

Four Student Responses:

Student 1. *40 cm*	Student 2. *40 cm because I added them and subtracted*
Student 3. *40 cm. Frill neck is 6 geckos so 6 X 8 = 48.* *Difference is 48 - 8 = 40*	Student 4. 

Scoring Rubric:

0	No response or irrelevant response
1	Correct (40 cm) but no reasoning or explanation provided
2	Correct, incomplete reasoning or an operational description given
3	Correct, correct reasoning using words, diagram or symbols

Using the Assessment Option

Distribute the booklets. Stress that the purpose of doing this is to inform future teaching – it is in the student's best interests to do as well as they can and not copy. Go through the instructions on the second page of the assessment booklet.

Encourage working – Students are expected record all of their work in the Assessment Option booklet so there is **no need** for scrap paper or jotters etc. Encourage students to explain their reasoning using words, diagrams or equations.

If they need additional space they should use the blank page at the back of the booklet. A single line should be placed through any rejected work (i.e. not obliterated or rubbed out) as it could provide some clues to students' thinking.

Student support – The object of the exercise is not that students get the right answer, but that they are given an opportunity to demonstrate what they actually do know and can do largely on their own.

Teachers can support students by answering questions without telling them what to do. Avoid providing so much support that students are able to complete the task with little understanding of what they are doing or why.

Teachers may:

- read the task to any student with reading problems
- scribe an oral explanation for students whose thinking may not otherwise be fairly represented
- explain unusual words as required.

Keep unfinished Option booklets in a safe place and ensure as far as possible that all students have an opportunity to attempt all tasks.

After using the Assessment Option

Collect booklets – Make sure that each student has created a unique identifier and has written this in the place provided on their booklet.

Mark student work – wherever possible work with colleagues to do this using the option-specific Scoring Rubrics (included with each Assessment Option). Record student scores on the Student Score Sheet, noting any interesting responses/observations in the comments column.

Match to LAF – When the marking is completed, the student's total score can be compared to the option-specific Raw Score Translator (included with each Assessment Option). This will assign the student's performance to a Zone in the Learning Assessment Framework for Multiplicative Thinking.

Note: There may be a small number of students who receive a zero score or a perfect score. Assuming this represents the best they can do, all that can be said about these students is that they are either below Zone 1 or above Zone 8.

> **Because this is a trial please complete the spreadsheet** provided with each student's score on each item. Use the Student Identifier only, but please indicate for each student their gender and year level. This information will help to refine the materials and determine their usefulness. Send this spreadsheet electronically to the project team.

Identify any student work that might provide useful work samples to help other teachers. These examples may be interesting, unusual or creative responses. The examples can be high or low level (the project needs a wide range of examples).

Make sure the Student Identifier is clearly marked on the work sample but remove any other identifying material (e.g., names etc). Photocopy or scan the work sample and return it to the project team.

Any other comments you may wish to make about the materials or the process, the Teacher Booklets and the student Assessment Tasks will be welcomed by the project team.

Where to next?

Refer to the teaching advice, that is, the **Learning Assessment Framework for Multiplicative Thinking (LAF)** below to determine a starting point for teaching and/or targeted intervention.

Student work samples

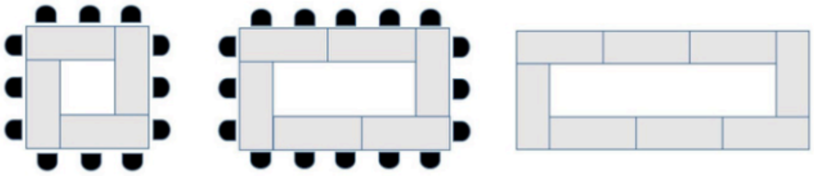
Because this is a trial we have yet to collect work samples specific to Options 3 and 4.

The following responses are taken from an earlier trial.

Board Room Tables ...

The diagrams below show how these tables can be arranged for different numbers of people.

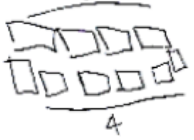
(No one sits inside the arrangement.)



Size 1	Size 2	Size 3
4 tables	6 tables	tables
12 people	16 people	people

[ABRT2]

How many tables are in a Size 4 arrangement? 10 tables

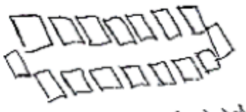


This item is scored on a 0,1 basis. The response is correct, so it is scored as a 1

[ABRT3]

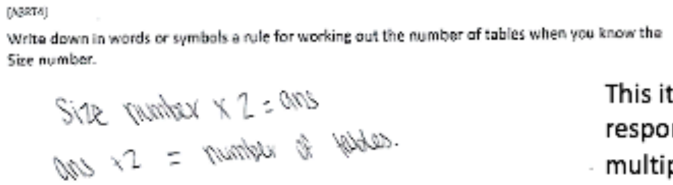
How many tables are in a Size 7 arrangement? 16 tables

Explain your reasoning.



There are 7 tables on each side (2x7=14) + the two tables at each end which connect them = 16.

This item is scored on a 0,1,2 basis. The response is scored as a 2 as although a diagram is not mentioned in the rubric, it is supported by words



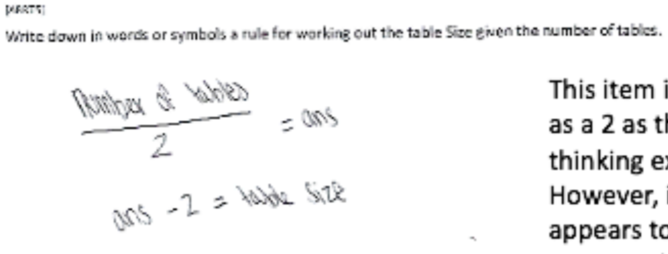
[A8RT4]

Write down in words or symbols a rule for working out the number of tables when you know the Size number.

Size number x 2 = ans

ans +2 = number of tables.

This item is scored on a 0,1,2 basis. The response is "correct, evidence of multiplicative thinking expressed in words (e.g., *two times the Size number plus two*) or in symbols (e.g., N = 2 x size + 2)" so it is scored as a 2.



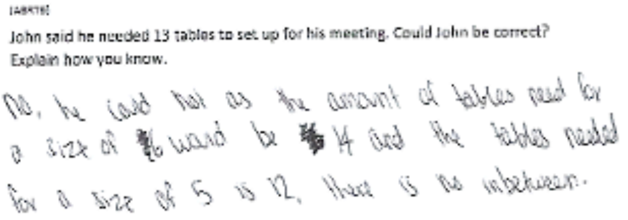
[A8RT5]

Write down in words or symbols a rule for working out the table Size given the number of tables.

$$\frac{\text{Number of tables}}{2} = \text{ans}$$

ans -2 = table Size

This item is scored on a 0,1,2 basis. It is scored as a 2 as there is evidence of multiplicative thinking expressed in words.
However, it is worth noting that this student appears to have difficulty constructing/expressing a pattern more formally

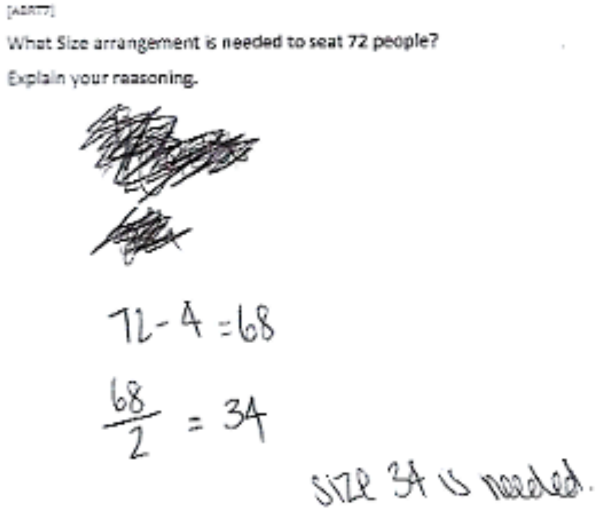


[A8RT6]

John said he needed 13 tables to set up for his meeting. Could John be correct?
Explain how you know.

No, he could not as the amount of tables need for
a size of 6 would be 14 and the tables needed
for a size of 5 is 12, there is no inbetween.

This item is scored on a 0,1,2,3 basis. It is scored as a 3 as it recognises the relationship between table size and number of tables



[A8RT7]

What Size arrangement is needed to seat 72 people?

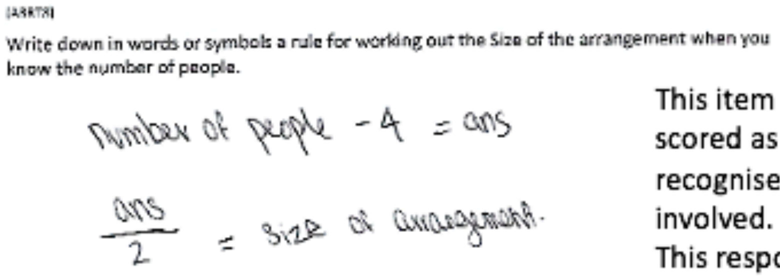
Explain your reasoning.

72 - 4 = 68

$$\frac{68}{2} = 34$$

size 34 is needed.

This item is scored on a 0,1,2.3 basis. It is scored as a 1 as although it is incorrect, it is not entirely irrelevant.
This response is worth investigating, it appears as though the student has used an inappropriate generalisation.
Note also, the two-step solution rather than the use of a pattern



[A8RT8]

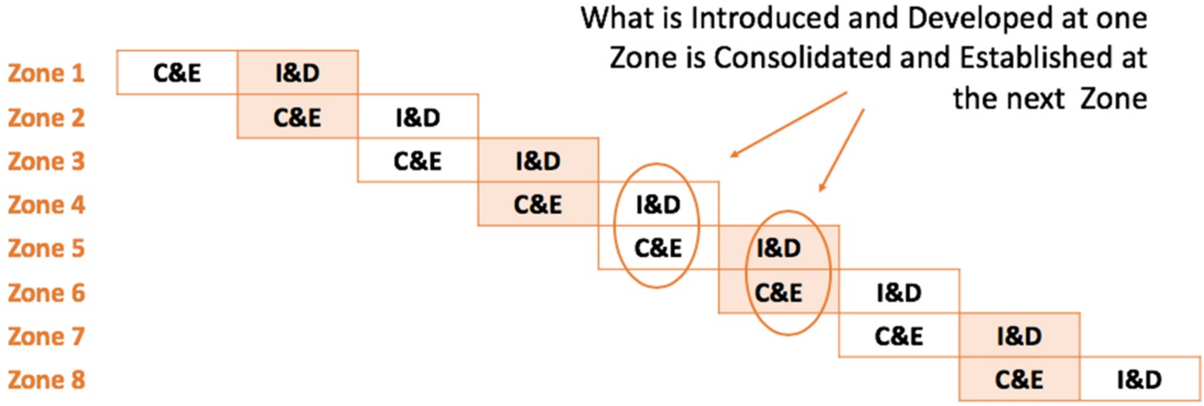
Write down in words or symbols a rule for working out the Size of the arrangement when you know the number of people.

number of people - 4 = ans

$$\frac{\text{ans}}{2} = \text{Size of arrangement.}$$

This item is scored on a 0,1,2.3 basis. It is scored as a 1 as although it is incorrect, it recognises that division and subtraction are involved.
This response is worth investigating for the same reasons as above

The Learning and Assessment Framework for Multiplicative Thinking (LAF)



What is Introduced and Developed at one Zone is Consolidated and Established at the next Zone

The LAF provides teaching advice to support a targeted teaching approach to multiplicative thinking. It should be used as the first 'port of call' in deciding how best to support student learning. As students are located at the point on the learning progression where they have a 50% chance of successfully completing the items at that level of difficulty, the advice for each Zone is presented in terms of what needs to be consolidated and established and what needs to be introduced and developed to scaffold students' progression to the next Zone.

It is important to note that the advice about what is introduced and developed at one Zone (e.g. Zone 4) is the same as the advice about what needs to be consolidated and established at the next Zone (e.g. Zone 5).

> Note for this Trial Version of the Manual – The LAF will be updated on the basis of the trial data. In the meantime, the LAF below contains a number of references to papers, presentations and tasks that were included in the original LAF, these can be found at https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/resourcelibrary.aspx

Support Material - the SNMY site contains a number of other resources from the original project, specifically, Learning Plans and Authentic Tasks. These have been extensively reviewed both as a result of the RMFII project and more recently as part of the Growing Mathematically project. These resources have now been updated and linked to exemplary resources such as reSolve and maths300. They are now available on the AAMT Growing Mathematically website as **Zone -Based Targeted Teaching Activities** (the link to the website will be made available shortly).

Zone Description	**Teaching Implications**
Zone 1 – Primitive Modelling Can solve simple multiplication and division problems involving relatively small whole numbers (e.g. *Butterfly House* parts *a* and *b*)*, but tends to rely on drawing, models and count-all strategies (e.g. draws and counts all pots for part *a* of *Packing Pots*). May use skip counting (repeated addition) for groups less than 5 (e.g. to find number of tables needed to seat up to 20 people in *Tables and Chairs*) Can make simple observations from data given in a task (e.g. *Adventure Camp a*) and reproduce a simple pattern (e.g. *Tables and Chairs a* to *e*) Multiplicative thinking (MT) not really apparent as no indication that groups are perceived as composite units, dealt with systematically, or that the number of groups can be manipulated to support a more efficient calculation * these items are from the original SNMY research	**Consolidate/establish:** **Trusting the count** for numbers to 10 (e.g. for 6 this involves working with mental objects for 6 without having to model and/or count-all). Use flash cards to develop **subitising** (i.e. ability to say how many without counting) for numbers to 5 initially and then to 10 and beyond using **part-part-whole knowledge** (e.g. 8 is 4 and 4, or 5 and 3 more, or 2 less than 10). Practice regularly **Simple skip counting** to determine how many in a collection and to establish numbers up to 5 as countable objects, for example, count by twos, fives and tens, using concrete materials and a 0-99 Number Chart **Mental strategies for addition and subtraction facts to 20** for example, *Count on from larger* (e.g. for 2 and 7, think, 7, 8, 9), *Double and near doubles* (e.g. use ten-frames and a 2-row bead-frame to show that 7 and 7 is 10 and 4 more, 14), and *Make-to-ten* (e.g. for 6 and 8, think, 8, 10, 14, scaffold using open number lines). Explore and name mental strategies to solve subtraction problems such as 7 take 2, 12 take 5, and 16 take 9. Practice (e.g. by using *Number Charts* from Maths 300) **2-digit place-value** – working flexibly with ones and tens by making, naming, recording, comparing, ordering, counting forwards and backwards in place-value parts, and renaming. Play the 'Place-Value Game' (see Siemon et al, 2015)[3]
	Introduce/develop: **Doubling (and halving) strategies** for 2-digit numbers that do not require renaming (e.g. 34 and 34, half of 46), build to numbers that require some additional thinking (e.g. to double 36, double 3 tens, double 6 ones, 60 and 12 ones, 72) **Extended mental strategies for addition and subtraction**, use efficient, place-value based strategies (e.g. 37 and 24, think: 37, 47, 57, 60, 61). Use open number lines to scaffold thinking **Efficient and reliable strategies for counting large collections** (e.g. count a collection of 50 or more by 2s, 5s or 10s) with a focus on how to organise the number of groups to facilitate the count (e.g. by arranging the groups systematically in lines or arrays and then skip counting) **How to make, name and use arrays/regions** to solve simple multiplication or sharing problems using concrete materials, and skip counting (e.g. 1 four, 2 fours, 3 fours …), leading to more efficient counting strategies based on reading arrays in terms of a consistent number of rows (e.g. 4 rows of anything, that is, 4 ones, 4 twos, 4 threes, 4 fours, …) **3-digit place-value** – working flexibly with tens and hundreds (by making with MAB, naming, recording, comparing, ordering, counting forwards and backwards in place-value parts, and renaming - see Siemon et al, 2015) **Strategies for unpacking and comprehending problem situations** (e.g. read and re-tell, ask questions such as, What is the question asking? What do we need to do? …). Use realistic word problems to explore different ideas for multiplication and division. For example, 3 rows, 7 chairs in each row, how many chairs (array)? Mandy has three times as many…as Tom…, how many … does she have (scalar idea)? 24 cards shared among 6 students, how many each (partition)? Lollipops cost 5c each, how much for 4 ('for each' idea)? **How to explain and justify** solution strategies orally and in writing through words and pictures (important for mathematical literacy)

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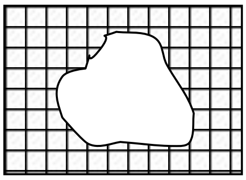
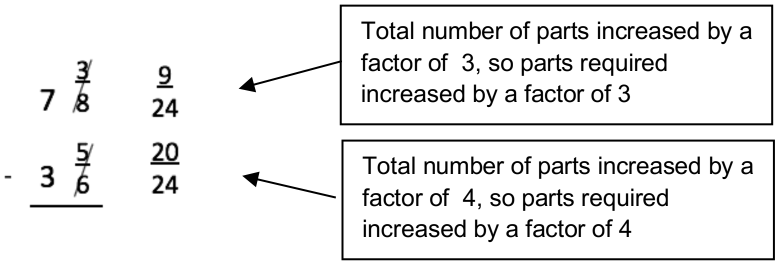
[3] Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R. & Warren, E. (2015). *Teaching Mathematics: Foundations to Middle Years.* Melbourne: Oxford University Press

Zone 2: Intuitive Modelling Trusts the count for groups of 2 and 5, that is, can use these numbers as units for counting (e.g. *Tables & Chairs j, Butterfly House d*), counts large collections efficiently, systematically keeps track of count (for instance may order groups in arrays or as a list) but needs to 'see' all groups (e.g. *Tiles, Tiles, Tiles a*, or for *Butterfly House e*, may use list and/or doubling as follows: 2 butterflies 5 drops 4 butterflies 10 drops 6 butterflies 15 drops … 12 butterflies 30 drops) Can share collections into equal groups/parts (e.g. *Pizza Party a* and *b*). Recognises small numbers as composite units (e.g. can count equal groups, skip count by twos, threes and fives) Recognises multiplication is relevant (e.g. *Packing Pots c, Speedy Snail a*) but tends not to be able to follow this through to solution Can list some of the options in simple Cartesian Product situations (e.g. *Canteen Capers a*) Orders 2-digit numbers (e.g. partially correct ordering of times in *Swimming Sports a*) Some evidence of multiplicative thinking as equal groups/shares seen as entities that can be counted systematically	**Consolidate/establish:** See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **More efficient strategies for counting groups** based on a **change in focus** from a count of equal groups (e.g. (1 three, 2 threes, 3 threes, 4 threes, …) to a consistent number of groups (e.g. 3 ones, 3 twos, 3 threes, 3 fours, …) which underpin the more efficient mental strategies listed below and ultimately lead to the **factor-factor-product idea** **Array/region-based mental strategies for multiplication facts to 100** for example, *doubling* (for 2s facts), *doubling and 1 more group* (for 3s facts), *double doubles* (for 4s facts), *relate to tens* (for 5s and 9s facts) and so on (see *There's More to counting Than Meets the Eye*) Efficient strategies for solving problems where **arrays** and **regions** only partially observed  For example, paint is spilled on a tiled floor. How many tiles to replace? How many altogether? How do you know? **Commutativity,** by exploring the relationship between arrays and regions such as 3 fours and 4 threes. Play 'Multiplication Toss' **Informal division strategies** such as *think of multiplication* and *halving,* (e.g. 16 divided by 4, think: 4 'whats' are 16? 4; or half of 16 is 8, half of 8 is 4) **Extended mental strategies for multiplication** (e.g. for 3 twenty-fives, Think: double 25, 50, and twenty-five more, 75) and use place-value based strategies such as 10 groups and 4 more groups for 14 groups **Simple proportion problems** involving non-numerical comparisons (e.g. If Nick mixed less cordial with more water than he did yesterday, his drink would taste (a) stronger, (b) weaker (c) exactly the same, or (d) not enough information to tell) **How to recognise and describe simple relationships** and patterns (e.g. 'double and add 2' from models, diagrams and tables; or notice that a diagonal pattern on a 0-99 chart is a count of 11, 1 ten and 1 ones) **Language of fractions** through practical experience with both **continuous and discrete, 'real-world' fraction models** for example, 3 quarters of the pizza, half the class), distinguish between how many and how much (e.g. in 2 thirds the numeral indicates how many, the name indicates how much) **Halving partitioning strategy**, through paper folding (kinder squares and streamers), cutting plasticine 'cakes' and 'pizzas', sharing collections equally (counters, cards etc), apply thinking involved to help children create their own fraction diagrams. Focus on making and naming parts in the halving family (e.g. 8 parts, eighths) including mixed fractions (e.g. "2 and 3 quarters") and informal recording (e.g. 3 eighths), no symbols **Key fraction generalisations** – that is, that equal parts are necessary and that the number of parts names the part

Zone 3: Sensing Demonstrates intuitive sense of proportion (e.g. partial solution to *Butterfly House f*) and partitioning (e.g. *Missing Numbers b*) Works with ‘useful’ numbers such as 2 and 5, and strategies such as doubling and halving (e.g. *Packing Pots b,* and *Pizza Party c*) May list all options in a simple Cartesian product situation (e.g. *Canteen Capers b*), but cannot explain or justify solutions Uses abbreviated methods for counting groups, for example, doubling and doubling again to find 4 groups of, or repeated halving to compare simple fractions (e.g. *Pizza Party c*) Beginning to work with larger whole numbers and patterns but tends to rely on count all methods or additive thinking to solve problems (e.g. *Stained Glass Windows a* and *b, Tiles, Tiles, Tiles b)*	**Consolidate/establish:** See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **Place-value based strategies** for informally solving problems involving single-digit by two-digit multiplication (e.g. for 3 twenty-eights, THINK, 3 by 2 tens, 60 and 24 more, 84) mentally or in writing Initial **recording to support place-value** for multiplication facts (see *Siemon et al, 2015* and *There’s More to Counting Than Meets the Eye*) More **efficient strategies for solving number problems involving simple proportion** (e.g. recognise as two-step problems, What do I do first? Find value for common amount. What do I do next? Determine multiplier/factor and apply. Why?) How **to rename number of groups** (e.g. think of 6 fours as 5 fours and 1 more four), Practice (e.g. by using ‘Multiplication Toss Game’*).* Re-name composite numbers in terms of equal groups (e.g. 18 is 2 nines, 9 twos, 3 sixes, 6 threes) **Cartesian product** or **for each** idea using concrete materials and relatively simple problems such as 3 tops and 2 bottoms, how many outfits, or how many different types of pizzas given choice of small, large, medium and 4 varieties? Discuss how to recognise problems of this type and how to keep track of the count such as draw all options, make a list or a table (tree diagrams appear to be too difficult at this level, these are included in Zone 5) How to **interpret problem situations and solutions relevant to context** (e.g. Ask, What operation is needed? Why? What does it mean in terms of original question?) Simple, practical division problems that require the **interpretation of remainders** relevant to context **Practical sharing situations that introduce names for simple fractional parts beyond the halving family** (e.g. thirds for 3 equal parts/shares, sixths for 6 equal parts etc) and help **build a sense of fractional parts**, for example, 3 sixths is the same as a half or 50%, 7 eighths is nearly 1, “2 and 1 tenth” is close to 2. Use a range of continuous and discrete fraction models including mixed fraction models **Thirding and fifthing partitioning strategies** through paper folding (kinder squares and streamers), cutting plasticine ‘cakes’ and ‘pizzas’, sharing collections equally (counters, cards etc), **apply thinking involved to help children create their own fraction diagrams (regions) and number line representations** (see Siemon (2004) *Partitioning – The Missing Link in building Fraction Knowledge and Confidence*. Focus on making and naming parts in the thirding and fifthing families (e.g. 5 parts, fifths) including mixed fractions (e.g. “2 and 5 ninths”) and informal recording (e.g. 4 fifths), no symbols. Revisit key fraction generalisations (see Level 2), include whole to part models (e.g. partition to show 3 quarters) and part to whole (e.g. if this is 1 third, show me the whole) and use diagrams and representations to **rename related fractions** **Extend partitioning strategies** to construct number line representations. Use multiple fraction representations **Key fraction generalisations** – equal parts, as the number of parts increase the size of the part gets smaller; the number of parts names the part (e.g., 8 parts, eighths) and the size of the part depends upon the size of the whole

Zone 4: Strategy Exploring Solves more familiar multiplication and division problems involving two-digit numbers (e.g. *Butterfly House c* and *d, Packing Pots c, Speedy Snail a*) Tend to rely on additive thinking, drawings and/or informal strategies to tackle problems involving larger numbers and/or decimals and less familiar situations (e.g. *Packing Pots d*, *Filling the Buses a* and *b*, *Tables & Chairs g* and *h*, *Butterfly House h* and *g, Speedy Snail c, Computer Game a, Stained Glass Windows a* and *b*). Tend not to explain their thinking or indicate working Able to partition given number or quantity into equal parts and describe part formally (e.g. *Pizza Party a* and *b*), and locate familiar fractions (e.g. *Missing Numbers a*) Beginning to work with simple proportion, for example, can make a start, represent problem, but unable to complete successfully or justify their thinking (e.g. *How Far a, School Fair a* and *b*)	**Consolidate/establish:** See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **More efficient strategies for multiplying and dividing larger whole numbers** independently of models (e.g. strategies based on: doubling, renaming the number of groups, factors, place-value, and known addition facts, for example, for dividing 564 by 8, THINK, 8 what's are 560? 8 by 7 tens or 70, so 70 and 4 remainder. for example, for 3908 divided by 10, RENAME as, 390 tens and 8 ones, so 390.8) **Tenths as a new place-value part**, by making/representing, naming and recording ones and tenths (see Siemon et al, 2015), consolidate by comparing, ordering, sequencing counting forwards and backwards in ones and/or tenths, and renaming How to partition continuous quantities more generally using the **halving, thirding, fifthing strategies** (see Siemon et al, 2015 and Siemon (2004) *Partitioning – The Missing Link in building Fraction Knowledge and Confidence*), for example, recognise that sixths can be made by halving and thirding (or vice versa), tenths can be made by fifthing and halving etc, use this knowledge to construct fraction diagrams (e.g. region models) and representations (e.g. number line) for common fractions and decimals including mixed numbers **Informal, partition-based strategies for renaming simple unlike fractions**, for example, recognise that thirds and fifths can be renamed by thirding and then fifthing (or vice versa) on a common diagram, for example,  Link to region model of multiplication (in this case 3 fives, or 3 parts by 5 parts) to recognise that thirds by fifths are fifteenths, so 2 thirds (2 rows) can be renamed as 10 fifths and 4 fifths (4 columns) can be renamed as 12 fifteenths. Use partitioning strategies to **informally add and subtract like and related fractions** **Key fraction generalisations** - that is, recognise that equal parts are necessary, the total number of parts names the part, and as the total number of parts increases they get smaller (this idea is crucial for the later development of more formal strategies for renaming fractions (see Level 5) which relate the number of parts initially (3, thirds) to the final number of parts (15, fifteenths) in terms of factors, that is, the number of parts has been increased by a factor of 5) **Metacognitive strategies** to support problem comprehension, problem representation, strategy monitoring/checking, and interpretation of outcomes relevant to context (see Siemon and Booker (1990) paper on *Teaching and Learning For, About and Through Problem Solving*) **Simple proportion problems that introduce techniques for dealing with these situations** (e.g. find for 1 then multiply or divide as appropriate, using scale diagrams and interpreting distances from maps)

Zone 5: Strategy Refining Systematically solves simple proportion and array problems (e.g. *Butterfly House e*, *Packing Pots a, How Far a*) suggesting multiplicative thinking. May use additive thinking to solve simple proportion problems involving fractions (e.g. *School Fair a, Speedy Snail b*) Able to solve simple, 2-step problems using a recognised rule/relationship (e.g. *Fencing the Freeway a*) but finds this difficult for larger numbers (e.g. *Tables & Chairs* k and *l, Tiles, Tiles, Tiles c, Stained Glass Windows c*) Able to order numbers involving tens, ones, tenths and hundredths in supportive context (*Swimming Sports a*) Able to determine all options in Cartesian product situations involving relatively small numbers, but tends to do this additively (e.g. *Canteen Capers* a, *Butterfly House l* and *i*) Beginning to work with decimal numbers and percent (e.g. *Swimming Sports a* and *b*, *Computer Game b*) but unable to apply efficiently to solve problems Some evidence that multiplicative thinking being used to support partitioning (e.g. *Missing Numbers b*) Beginning to approach a broader range of multiplicative situations more systematically	**Consolidate/establish:** See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **Place-value** ideas and strategies for 5 digits and beyond if not already developed and decimal fractions to hundredths (see partitioning below) including renaming **Flexible, meaningful and efficient strategies** for multiplying and dividing by multiples of ten (e.g. 2.13 by 10, THINK, 21 ones and 3 tenths, 21.3) The **area idea** to support multi-digit multiplication and formal recording (see Siemon et al, 2015) and more efficient strategies for representing and solving an **expanded range of Cartesian product problems** involving three or more variables and **tree diagram representations** **Formal terminology** associated with multiplication and division such as factor, product, divisor, multiplier and raised to the power of …. Play 'Factor Cross Game'. Use calculators to explore what happens with repeated factors, for example, 4 x 4 x 4 x 4 …, factors less than 1, and negative factors. **Informal, partition-based strategies for renaming an expanded range of unrelated fractions** as a precursor to developing an efficient, more formal strategy for generating equivalent fractions (see below), for example, explore using paper folding, diagrams and line models how sixths and eighths could be renamed as forty-eighths but they can also be renamed as twenty-fourths because both are factors of 24 The **generalisation for renaming fractions**, that is, if the number of equal parts (represented by the denominator) increases/decreases by a certain factor then the number of parts required (indicated by the numerator) increases/decreases by the same factor **Written solution strategies for the addition and subtraction of unlike fractions**, for example, think of a diagram showing sixths by eighths … forty-eighths… Is this the simplest? No, twenty-fourths will do, rename fractions by inspection  Total number of parts increased by a factor of 3, so parts required increased by a factor of 3 Total number of parts increased by a factor of 4, so parts required increased by a factor of 4 9 twenty-fourths can't take 20 twenty-fourths, trade 1 one for 24 twenty-fourths to get 6 and 33 twenty-fourths, subtraction is then relatively straightforward Explore **link between multiplication and division and fractions** including decimals (e.g. 3 pizzas shared among 4, 3 divided by 4 is 0.75 etc) to understand **fraction as operator idea** (e.g. ¾ of 120, 75% of $48, 250% of 458,239). Use 'Multiple Patterns Worksheet' (See Support Materials). Establish benchmark equivalences (e.g. 1 third =$33\frac{1}{3}$ %) **Metacognitive strategies** to support problem comprehension, strategy monitoring/checking, and interpretation of outcomes relevant to context (see Siemon and Booker (1990) paper on *Teaching and Learning For, About and Through Problem Solving*)

Zone 6: Strategy Extending	**Consolidate/establish:**
Can work with Cartesian Product idea to systematically list or determine the number of options (e.g. *Canteen Capers b*, *Butterfly House i* and *h*) Can solve a broader range of multiplication and division problems involving two digit numbers, patterns and/or proportion (e.g. *Tables & Chairs h*, *Butterfly House f, Stained Glass Windows b* and *c, Computer Game a and b*) but may not be able to explain or justify solution strategy (e.g. *Fencing the Freeway b, Fencing the Freeway d,* and *Swimming Sports b, How Far b, Speedy Snail b*) Able to rename and compare fractions in the halving family (e.g. *Pizza Party* c) and use partitioning strategies to locate simple fractions (e.g. *Missing Numbers* a) Developing sense of proportion (e.g. sees relevance of proportion in *Adventure Camp b*, *Tiles, Tiles, Tiles b*), but unable to explain or justify thinking Developing a degree of comfort with working mentally with multiplication and division facts	See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **Hundredths as a new place-value part**, by making/representing, naming and recording ones, tenths, and hundredths (see Siemon et al, 2015), consolidate by comparing, ordering, sequencing counting forwards and backwards in place-value parts, and renaming. Link to % How to **explain and justify solution strategies** for problems involving multiplication and division (see Multiplication Workshop handout in Support Materials), particularly in relation to interpreting decimal remainders appropriate to context, for example, How many buses will be needed to take 594 students and teachers to the school Speech night, assuming each bus hold 45 passengers and everyone must wear a seatbelt? More **efficient, systematic, and/or generalizable processes** for dealing with **proportion problems** (e.g. use of the 'for each' idea, formal recording, and the use of fractions, percent to justify claims), for example, Jane scored 14 goals from 20 attempts. Emma scored 18 goals from 25 attempts. Which girl should be selected for the school basketball team and why? 6 girls share 4 pizzas equally. 8 boys share 6 pizzas equally. Who had more pizza, the girls or the boys? 35 feral cats were found in a 146 hectare nature reserve. 27 feral cats were found in a 103 hectare reserve. Which reserve had the biggest feral cat problem? Orange juice is sold in different sized containers: 5L for \$14, 2 L for \$5, and 500mL for \$1.35. Which represents the best value for money? **More efficient strategies and formal processes** for working with multiplication and division involving larger numbers based on sound place-value ideas, for example, 3486 x 21 can be estimated by thinking about 35 hundreds by 2 tens, 70 thousands, and 1 more group of 35 hundred, that is, 73,500, or it can be calculated by using factors of 21, that is, 3486 x 3 x 7. Two-digit multiplication can be used to support the multiplication of ones and tenths by ones and tenths, for example, for 2.3 by 5.7, rename as tenths and compute as 23 tenths by 57 tenths, which gives 1311 hundredths hence 13.11. Consider a broader range of problems and applications, for example, Average gate takings per day over the World Cricket cup Series Matt rode around the park 8 times. The odometer on his bike indicated that he ridden a total of 15 km. How far was it around the park? After 11 training sessions, Kate's average time for 100 metres butterfly was 61.3 seconds. In her next 2 trials, Kate clocked 61.21 and 60.87 seconds. What was her new average time? **Integers** using real-world examples such as heights above and below sea-level, temperatures above and below zero, simple addition and difference calculations The **notion of variable** and how to **recognise and formally describe patterns** involving all four operations. Use 'Max's Matchsticks' to explore how patterns may be viewed differently leading to different ways of counting and forms of representation.

Zone 7: Connecting	**Consolidate/establish:**
Able to solve and explain one-step problems involving multiplication and division with whole numbers using informal strategies and/or formal recording (e.g. *Filling the Buses a, Fencing the Freeway d, Packing Pots d*) Can solve and explain solutions to problems involving simple patterns, percent and proportion (e.g. *Fencing the Freeway c, Swimming Sports b, Butterfly House g, Tables & Chairs g* and *l, Speedy Snail c, Tiles, Tiles, Tiles b and c, School Fair a, Stained Glass Windows a, Computer Game b, How Far b*). May not be able to show working and/or explain strategies for situations involving larger numbers (e.g. *Tables & Chairs m* and *k, Tiles, Tiles, Tiles c*) or less familiar problems (e.g. *Adventure Camp b, School Fair b, How Far c*) Locates fractions using efficient partitioning strategies (e.g. *Missing Numbers a*) Beginning to make connections between problems and solution strategies and how to communicate this mathematically	See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **Strategies for comparing, ordering, sequencing, counting forwards and backwards in place-value parts, and renaming** large whole numbers, common fractions, decimals, and integers (e.g. a 3 to 4 metre length of rope, appropriately labelled number cards and pegs could be used to sequence numbers from 100 to 1,000,000, from -3 to +3, from 2 to 5 and so on). The metaphor of a magnifying glass can be used to locate numbers involving hundredths or thousandths on a number line as a result of successive *tenthing* (see Siemon et al, 2015 and Siemon (2004) *Partitioning – The Missing Link in building Fraction Knowledge and Confidence*) An appreciation of **inverse and identity relations,** for example, recognise which number when added leaves the original number unchanged (zero) and how inverses are determined in relation to this, for example, the inverse of 8 is -8 as -8 + 8 = 0 and 8 + -8 = 0. In a similar fashion, recognise that 1 is the corresponding number for multiplication, where the inverse of a number is defined as its **reciprocal**, for example, the inverse of 8 is $\frac{1}{8}$ **Index notation** for representing multiplication of repeated factors, for example, $5 \text{ x } 5 \text{ x } 5 \text{ x } 5 \text{ x } 5 \text{ x } 5 = 5^6$ A **more generalised understanding of place-value** and the structure of the number system in terms of exponentiation, for example, 10^{-3}, 10^{-2}, 10^{-1}, 10^0, 10^1, 10^2 10^3 … **Strategies to recognise and apply** multiplication and division in a broader range of situations including ratio, proportion, and unfamiliar, multiple-step problems, for example, *Orange Juice* task (see Support Materials) How to **recognise and describe number patterns more formally** for example, triangular numbers, square numbers, growth patterns (e.g. 'Garden Beds' from *Maths 300* and 'Super Market Packer' from Support Materials) **Notation to support general arithmetic** (simple algebra), for example, recognise and understand the meaning of expressions such as $x+4$, $3x$, $5x^2$, or $\frac{x-1}{3}$ **Ratio as the comparison of any two quantities**, for example, the comparison of the number of feral cats to the size of the national park. Recognise that ratios can be used to compare measures of the same type (e.g. the number of feral cats compared to the number of feral dogs) and that within this, two types of comparison are possible, for instance, one can compare the parts to the parts (e.g. cats to dogs) or the parts to the whole (e.g. cats to the total number of cats and dogs). Ratios can be also used to compare measures of different types, when used this way they are referred to as rates (e.g. the number of feral cats per square kilometre). Ratios are not always rational numbers (e.g. the ratio of the circumference of a circle to its diameter) **Strategies for recognising and representing proportion problems involving larger numbers and/or fractions** (e.g. problems involving scale such as map calculations, increasing/reducing ingredients in a recipe, and simple problems involving derived measures such as volume, density, speed, and chance)

Zone 8: Reflective Knowing Can use appropriate representations, language and symbols to solve and justify a wide range of problems involving unfamiliar multiplicative situations including fractions and decimals (e.g. *Adventure Camp b*, *Speedy Snail b*) Can justify partitioning (e.g. *Missing Numbers b*) Can use and formally describe patterns in terms of general rules (e.g. *Tables and Chairs*, *m* and *k*) Beginning to work more systematically with complex, open-ended problems (e.g. *School Fair b, Computer Game c*)	**Consolidate/establish:** See the **Ideas and strategies** introduced/developed in the previous Zone
	Introduce/develop: **A broader range of multiplicative situations** for example, problems involving the calculation of area or volume, derived measures and rates, variation, complex proportion, and multiple step problems involving large whole numbers, decimals and fractions, for example, Find the volume of a cylinder 4 cm in diameter and 9 cm long. Find the surface area of a compound shape Foreign currency calculations Determine the amount of water lost to evaporation from the Hume Weir during the summer. **Strategies for simplifying expressions** for example, adding and subtracting like terms, and justifying and explaining the use of cancellation techniques for division through the use of common factors, for example, $\underline{42a} = 6a$ because $\frac{42a}{7} = \frac{7 \times 6a}{7}$ and $\frac{7}{7} = 1$ **Algebraic reasoning and representation strategies** to solve problems involving multiplicative relationships, for example, If 2 T-shirts and 2 drinks cost $44 and 1 T-shirt and 3 drinks cost $30, what is the price of each? 5 locker keys are returned at random to the students who own them. What is the probability that each student will receive the key that opens their locker? A mad scientist has a collection of beetles and spiders. The sensor in the floor of the enclosure indicated that there were 174 legs and the infra-red image indicated that there were 26 bodies altogether. How many were beetles and how many were spiders? 365 is an extraordinary number. It is the sum of 3 consecutive square numbers and also the sum of the next 2 consecutive square numbers. Find the numbers referred to. **Strategies for working with numbers and operations expressed in exponent form**, for example, why $2^3 \text{ x } 2^6 = 2^9$, investigate the structure of the place value system in terms of positive and negative powers of 10 **Explore non-linear, exponential situations** such as growth and decay (e.g. Radioactivity activity from maths300) **Writing mathematically** using appropriate symbolic text, using equivalent sentences to systematically arrive at a solution **More abstract problem solving situations** requiring an appreciation of problem solving as a process, the value of recognising problem type, and the development of a greater range of strategies and representations (e.g. tables, symbolic expressions, rule generation and testing) including the manipulation of symbols

Case Studies

The following case studies are taken from the Reframing Mathematical Futures projects, the Priority project in 2013 (RMF-P) and the subsequent RMF II project in 2015-2018. They are adapted from the ones reported in Siemon, Banks, & Prasad (2018).

Case Study 1

Palberton Middle School (a pseudonym) is located in a growing, outer suburb of a northern Australian city. At the time, the school had 560 Year 7 to 9 students from a diverse range of cultural backgrounds. The school leadership team and the maths staff were keen to improve mathematics learning outcomes so 'jumped at the chance' to participate in the RMF-P project as they could see this working well with their commitment to team teaching and using data to inform teaching approaches. After attending the initial workshop in Melbourne in July 2013, the specialist teacher in consultation with the school leadership and two other maths staff decided to target four of the Year 8 classes (50% of the cohort) in what remained of the 2013 school year.

The school's purpose-built accommodation facilitated team teaching approaches. Four classrooms were grouped around a central covered space with large sliding doors providing access to the central space from each classroom. Co-teaching arrangements were formalised in 'hub' agreements and the team co-taught two of the four classes while a parallel team of English teachers co-taught the other two classes. Learning support staff were available on most occasions to support the work of the co-teaching teams. The school timetable provided five 50-minute lessons per week for maths (and English) which included one double lesson.

The RMF team as it became known at the school, administered, marked and moderated SNMY Option 1 for the four Year 8 classes in August 2013 and created profiles for all students. The specialist shared the data with the school leadership team and a key figure in the Department of Education, who were "shocked" to see that 53% of the Year 8s assessed were in Zones 1 to 3 of the LAF. When the leadership group recognised what this meant, further in-kind resources were made available to support the work of the project.

A decision was made to use the double period in maths each week to implement a targeted teaching approach to multiplicative thinking. These lessons, which came to be referred to as RMF Maths, were structured to include a Do Daily session, an open-ended problem related to the mathematics being considered in the other three lessons, work in Zone groups on targeted teaching activities, and a formal period of reflection. The approximate time spent on each of these components was 10, 40, 40 and 10 minutes respectively. Each member of the team was responsible for two to three Zones. The team met weekly to plan Zone activities, many of which they adapted to be age-appropriate and met again on Saturdays for professional sharing and forward planning. The students were given project books which they decorated in which to record their reflections at the end of each RMF lesson. A template was provided to help structure the reflections. The booklets were collected and reviewed by the team who used the reflections to inform their planning before being returned at the beginning the next lesson with written. The team observed considerable changes in the nature and amount of reflective comments provided by the students over the course of the semester – the students looked forward to reading the feedback from the teachers and quickly settled in order to see what was written.

The targeted teaching activities and materials were organised and stored by Zone in the hub area in open shelving that was available to students. This enabled some level of choice if students wanted to move on to another activity or try an activity from another Zone. While this was a massive effort, the teachers felt it was worth as one of the first things they noticed were that there were far fewer instances of challenging behaviour to deal with and students were asking if they could do 'RMF maths' all of the time. Another positive outcome was that students were becoming more metacognitive in their responses to problems they were doing in the non-RMF lessons, for instance, the team noticed that many of the students started to explain their reasoning without being asked. Although the demands on the teaching staff were high with many additional hours per week spent on preparing and adapting Zone activities, the teachers felt that they had grown as a team and were more knowledgeable about how to deal with student misconceptions.

SNMY Option 2 was administered in November and marked and moderated by the co-teaching team. The data were de-identified, recorded on a spreadsheet and forwarded to the research team for analysis. Data from 70 matched pairs were available for analysis, the results of which can be seen in Figure 3. The improvement in multiplicative thinking was impressive with an adjusted effect size of 1.18.

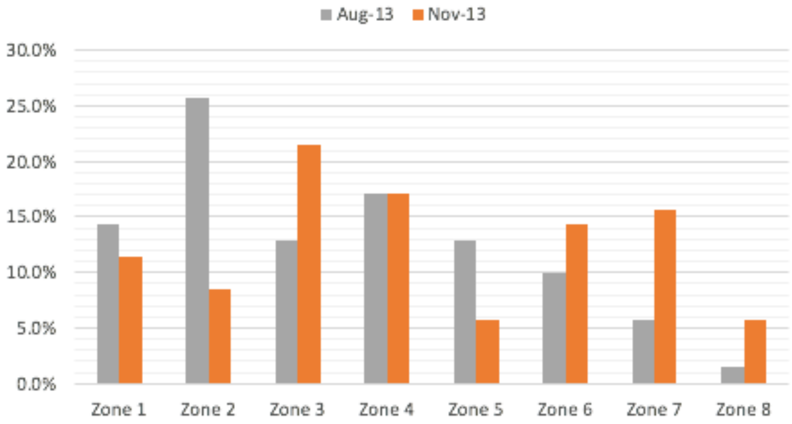


Figure 3. Proportion of Year 8 students by LAF Zone in August and November (n = 70).

Case Study 2

Plumpton High School (name used with permission) is located in an established western suburb of Sydney. It is a large multi-cultural 7 to 12 secondary school, a key goal of which is to "put students first". Plumpton High School came to use the SNMY materials and implement a targeted teaching approach to multiplicative thinking as a result of the school's participation in RMFII which was aimed at building a similar evidenced-based framework for mathematical reasoning in Years 7 to 10. As one of the 'new' schools that were unfamiliar with the notion of a targeted teaching, the school was asked to use the SNMY materials to identify and respond to student learning needs in relation to multiplicative thinking before contributing to the trial of the mathematical reasoning materials.

When offered the opportunity to participate in RMFII project in late 2014, the mathematics results at the school was a concern and number of students electing to pursue the more advanced maths courses in the senior years was declining. The school felt a change was needed. As a result, the school leadership not only agreed to participate in the project they decided to send an additional teacher to the initial three-day workshop in Melbourne in November 2014 at the school's expense that introduced teachers from the 'new' schools to multiplicative thinking and the SNMY materials. The RMF-P specialists who were continuing in the follow up project were able to share their SNMY results and describe what worked and what did not work in implementing a targeted teaching approach to multiplicative thinking in secondary school contexts. Key strategies that were variously adopted by the RMF-P schools that were most successful included team teaching, dedicated lesson times for targeted teaching and Zone-based activities, locally available resources, team planning time, additional time release, access to professional learning opportunities, and support of school leadership. RMF-P schools implemented these and other strategies to different extents and in different ways appropriate to their circumstances but the teachers from the 'new' schools such as Plumpton were able to draw on this information to plan how they would implement a targeted teaching approach.

On returning to school, a decision was made to focus on the whole of Year 8 in 2015. Teaching staff felt that the current Year 7 students would most benefit from the intervention and as they were still at school it would make sense to administer SNMY Option 1 in December of 2014. The school leadership supported the decision to focus on Year 8 in 2015 as this cohort would sit the NAPLAN test in Year 9 in 2016 which would provide an independent evaluation of the intervention.

In 2015, each of the six Year 8 classes had a separate 75-minute RMF lesson per week. During this time, the students worked in their Zone groupings initially on activities from the project Dropbox and/or ones prepared by the specialist. The specialist and one other of the senior maths teachers, dropped by the classrooms whenever they could to help and prepared resources in their free periods. As time went on and the demand for new, age-appropriate activities increased, the Year 8 teachers also developed and shared Zone-based activities with their colleagues. One of the ways in which this happened was at the Wednesday lunches, where Year 8 staff talked about what they were doing, reflected on progress and developed new ideas. A lesson template was developed and staff would workshop new lessons prior to delivery. Referred to as 'Live in Lessons', this enabled the team to iron out any potential issues and to make links to regular classroom teaching activities and content.

While project funding was provided to support the implementation of a targeted teaching approach to multiplicative thinking in four classes, the school decided to implement this approach in 6 classes of 30 students, which meant resources were tight. Priority was given to purchasing concrete materials and a separate area was set up to keep class booklets, resources and activities for easy collection and distribution. Initially, there was little buy-in from students and teachers as working in groups was something new for many. The existing class structure (semi-streamed) helped manage the targeted teaching approach but there was considerable variation in each classroom. Planning was essential and proved to be a key factor to the school's success. Over the course of the year, teachers found that they were incorporating many of Zone type activities into the curriculum being taught in the week, placing particular emphasis on the need to explain and justify solution strategies as this had proved to be a major sticking point early on. The team learnt as they went and kept on sharing, adjusting and implementing strategies/activities which worked in other classes. Staff meetings on Mondays were focussed on developing teachers' capacity to share resources and ideas to help the growth of targeted teaching in classrooms.

Gradually, everything became easier, the students were more accustomed to working in groups and appreciated the opportunity to experience success. Student engagement increased and the quality of their responses to school-based assessments improved noticeably. Teaching staff were more inclined to design reasoning activities for regular classroom teaching and provide time for students to apply what they know in unfamiliar contexts and marking rubrics were slowly incorporated into classroom assessment tasks.

SNMY Option 2 was administered, marked and moderated by Year 8 teachers and the two specialists in September 2015. The results were again de-identified and forwarded to the research team for analysis. The results were impressive and immediately bought buy in from senior management and other maths teachers. Additional teacher release was provided to support the preparation of resources, marking and moderating of assessments, and training of other staff members. The growth is shown in Figure 4 and represents an effect size of over 3.5.

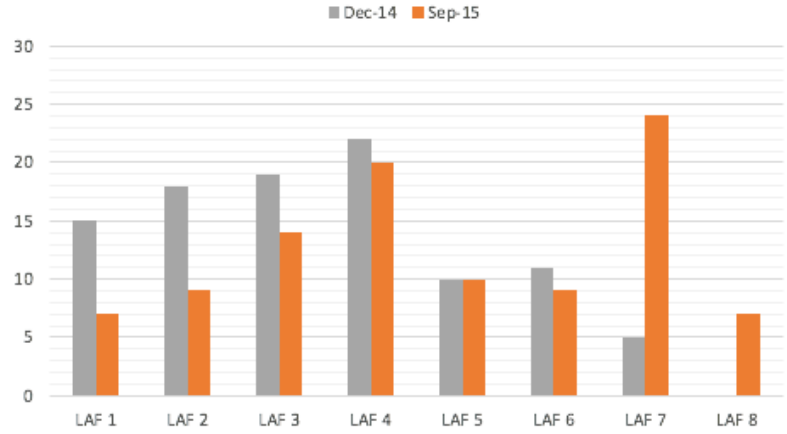


Figure 4. Proportion of Year 8 students by LAF Zone (141<n<152)

While not the only measure of success in school mathematics, the Year 9 NAPLAN results for the same cohort in 2016 provide conclusive evidence that targeted teaching makes a difference. Compared to the previous Year 9 who sat the NAPLAN test in 2015, the average scaled growth score for the school went from below all State in 2015 (45.6) to above all State in 2016 (51.1). But perhaps more telling are the respective growth comparisons between 2015 and 2016 of the proportion of students in the less than expected growth category versus the proportion of students in the greater than or equal to expected growth category.

Table 2. Growth comparisons for the 2015 and 2016 Year 9 cohorts, Plumpton High School.

	Less than Expected growth	Greater than or equal to expected growth
2015	48.7%	51.2%
2016	34.5%	65.5%

Mapping the LAF Zones to the *Australian Curriculum: Mathematics*

Midway through the RMFII project, feedback from research school teachers on the use of the SNMY materials suggested that it would be helpful to show how the evidenced-based *Learning Assessment Framework for Multiplicative Thinking* related to the *Australian Curriculum: Mathematics*. The table below was prepared by a member of the RMFII research team to address this need. The column on the left summarises key aspects of the LAF Zones. The column on the right lists the related content descriptors of the *Australian Curriculum: Mathematics* using colour coding.

LAF ZONES (Siemon et al., 2006)	LINKS TO THE *AUSTRALIAN CURRICULUM: MATHEMATICS*
Zone 1: • **Solves simple multiplication and division problems** involving relatively small whole numbers, but tends to rely on drawing, models and count-all strategies. • **May use skip counting for groups less than five.** • **Makes simple observations from data** and **extends simple number patterns**. • Multiplicative thinking (MT) not really apparent as no indication that groups are perceived as composite units, dealt with systematically, or that the number of groups can be manipulated to support more efficient calculation.	**Foundation Year:** • **Subitise small collections of objects** (ACMNA003) • Represent practical situations to model addition and **sharing** (ACMNA289) ***Problem Solving:*** ***use familiar counting sequences to solve unfamiliar problems.*** **Year 1:** • Develop confidence with number sequences to and from 100 by ones from any starting point. **Skip count by twos, fives and tens starting from zero** (ACMNA012) • **Investigate and describe number patterns** formed by **skip counting** and patterns with objects (ACMNA018) • Recognise, model, read, write and order numbers to at least 100. **Locate these numbers on a number line** (ACMNA013) • **Recognise and describe one-half as one of two equal parts of a whole.** (ACMNA016) ***Problem Solving:*** ***use familiar counting sequences to solve unfamiliar problems.*** **Year 2:** • **Describe patterns with numbers and identify missing elements** (ACMNA035) • **Recognise and represent division as grouping into equal sets and solve simple problems using these representations** (ACMNA032)
Zone 2: • **Counts large collections efficiently** – keeps track of count but needs to see all groups. • **Shares collections equally.** • **Recognises small numbers as composite units (e.g. can count equal groups, skip count by twos, threes and fives).** • **Recognises multiplication needed** but tends not to be able to follow this through to solution. •	**Year 2:** • **Group, partition and rearrange collections up to 1000 in hundreds, tens and ones to facilitate more efficient counting** (ACMNA028) • **Recognise and represent multiplication as repeated addition, groups and arrays** (ACMNA031) • **Recognise and interpret common uses of halves, quarters and eighths of shapes and collections** (ACMNA033)

• Lists some of the options in simple Cartesian product situations. • Some evidence of MT as equal groups/shares seen as entities that can be counted.	***Understanding:*** connecting number calculations with counting sequences and partitioning and combining numbers flexibly. ***Fluency:*** counting numbers in sequences readily. **Year 3:** • Investigate the conditions required for a number to be odd or even and identify odd and even numbers (ACMNA051) • Model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole (ACMNA058) • Represent and solve problems involving multiplication using efficient mental and written strategies and appropriate digital technologies (ACMNA057) • Recall multiplication facts of two, three, five and ten and related division facts (ACMNA056) ***Understanding:*** partitioning and combining numbers flexibly and representing unit fractions ***Fluency:*** recalling multiplication facts
Zone 3: • Demonstrates intuitive sense of proportion. • Works with useful numbers such as 2 and 5 and intuitive strategies to count/compare groups (e.g., doubling, or repeated halving to compare simple fractions). • May list all options in a simple Cartesian product, but cannot explain or justify solutions. • Beginning to work with larger whole numbers and patterns but tends to rely on count all methods or additive thinking (AT).	**Year 4:** • Investigate and use the properties of odd and even numbers (ACMNA071) • Apply place value to partition, rearrange and regroup numbers to at least tens of thousands to assist calculations and solve problems (ACMNA073) • Investigate number sequences involving multiples of 3, 4, 6, 7, 8, and 9 (ACMNA074) • Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075) • Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder (ACMNA076) • Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation (ACMNA079) • Explore and describe number patterns resulting from performing multiplication (ACMNA081) ***Problem Solving:*** *using properties of numbers to continue patterns* ***Reasoning:*** *using generalising from number properties and results of calculations and deriving strategies for unfamiliar multiplication and division tasks.*

Zone 4: - **Solves simple multiplication and division problems involving two-digit numbers.** - Tends to rely on AT, drawings and/or informal strategies to tackle problems involving larger numbers, decimals and/or less familiar situations. - Tends not to explain thinking or indicate working. - **Partitions given number or quantity into equal parts and describes part formally.** - Beginning to work with simple proportion.	**Year 4:** - **Investigate equivalent fractions used in contexts** (ACMNA077) - **Count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line** (ACMNA078) - **Solve word problems by using number sentences involving multiplication or division where there is no remainder** (ACMNA082) ***Understanding:*** ***partitioning*** *and* ***combining numbers flexibly*** **Year 5:** - **Identify and describe factors and multiples of whole numbers and use them to solve problems** (ACMNA098) - **Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies** (ACMNA100) - **Solve problems involving division by a one digit number, including those that result in a remainder** (ACMNA101) - **Compare and order common unit fractions and locate and represent them on a number line** (ACMNA102) - **Use equivalent number sentences involving multiplication and division to find unknown quantities** (ACMNA121) ***Understanding:*** ***comparing and ordering fractions and decimals and representing them in various ways*** ***Problem Solving:*** ***formulating and solving authentic problems using whole numbers***
Zone 5: - Solves whole number proportion and array problems systematically. - **Solves simple, 2-step problems using a recognised rule/relationship but finds this difficult for larger numbers.** - Determines all options in Cartesian product situations involving relatively small numbers but tends to do this additively. - **Beginning to work with decimal numbers and percent.** - **Some evidence MT being used to support partitioning.** - **Beginning to approach a broader range of multiplicative situations more systematically**	**Year 5:** - **Use efficient mental and written strategies and apply appropriate digital technologies to solve problems** (ACMNA291) - **Compare, order and represent decimals** (ACMNA105) ***Reasoning:*** ***investigating strategies to perform calculations efficiently*** *and* ***continuing patterns involving fractions and decimals*** **Year 6:** - **Identify and describe properties of prime, composite, square and triangular numbers** (ACMNA122)

	• **Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers** (ACMNA123) • **Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies** (ACMNA127) • **Multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies** (ACMNA129) • **Multiply and divide decimals by powers of 10** (ACMNA130) • **Make connections between equivalent fractions, decimals and percentages** (ACMNA131) • **Investigate and calculate percentage discounts of 10%, 25% and 50% on sale items, with and without digital technologies** (ACMNA132) • **Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence** (ACMNA133) ***Fluency:*** *calculating simple percentages, converting between fractions and decimals, and using operations with fractions, decimals and percentages* ***Problem Solving:*** *formulating and solving authentic problems using fractions, decimals and percentages*
Zone 6: • Systematically lists/determines the number of options in Cartesian product situation. • **Solves a broader range of multiplication and division problems** involving 2-digit numbers, patterns and/or proportion but may not be able to explain or justify solution strategy. • **Renames and compares fractions in the halving family, uses partitioning strategies to locate simple fractions.** • Developing sense of proportion, but unable to explain or justify thinking. • **Developing capacity to work mentally with multiplication and division facts**	**Year 6:** • **Compare fractions with related denominators and locate and represent them on a number line** (ACMNA125) ***Understanding:*** *representing fractions and decimals in various ways and describing connections between them* ***Reasoning:*** *explaining mental strategies for performing calculations* **Year 7:** • **Investigate index notation and represent whole numbers as products of powers of prime numbers** (ACMNA149) • **Investigate and use square roots of perfect square numbers** (ACMNA150) • **Apply the associative, commutative and distributive laws to aid mental and written computation** (ACMNA151) • **Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line** (ACMNA152) • **Multiply and divide fractions and decimals using efficient written strategies and digital technologies** (ACMNA154)

	• **Express one quantity as a fraction of another, with and without the use of digital technologies** (ACMNA155) • **Connect fractions, decimals and percentages and carry out simple conversions** (ACMNA157) • **Find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies.** (ACMNA158) • **Recognise and solve problems involving simple ratios** (ACMNA173) ***Fluency***: ***calculating accurately with integers*** *and* ***representing fractions and decimals in various ways*** ***Problem Solving***: ***formulating and solving authentic problems using numbers***
Zone 7: • **Solves and explains one-step problems involving multiplication and division with whole numbers using informal strategies and/or formal recording.** • **Solves and explains solutions to problems involving simple patterns, percent and proportion.** • May not be able to show working and/or explain strategies for situations involving larger numbers or less familiar problems. • **Constructs/locates fractions using efficient partitioning strategies.** • **Beginning to make connections between problems and solution strategies** and how to communicate this mathematically	**Year 7:** • **Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line** (ACMNA152) ***Understanding: describing patterns in uses of indices with whole numbers,*** *and* ***connecting the laws and properties of numbers to algebraic terms and expressions*** ***Fluency: calculating accurately with integers*** *and* ***representing fractions and decimals in various ways*** ***Problem Solving: formulating and solving authentic problems using numbers*** ***Reasoning***: ***applying the number laws to calculations*** *and* ***applying an understanding of ratio*** **Year 8:** • **Use index notation with numbers to establish the index laws with positive integral indices and the zero index** (ACMNA182) • **Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies** (ACMNA183) • **Solve problems involving the use of percentages, including percentage increases and decreases,** with and without digital technologies (ACMNA187) • **Solve a range of problems involving rates and ratios,** with and without digital technologies (ACMNA188) • **Simplify algebraic expressions involving the four operations** (ACMNA192) ***Understanding: identifying commonalities between operations with algebra and arithmetic.***

Zone 8: - **Uses appropriate representations, language and symbols to solve and justify a wide range of problems involving unfamiliar multiplicative situations,** fractions and decimals. - Can justify **partitioning**, and formally **describe patterns in terms of general rules.** - Beginning to **work more systematically** with complex, open-ended problems.	**Year 8:** - **Extend and apply the distributive law to the expansion of algebraic expressions** (ACMNA190) - **Factorise algebraic expressions by identifying numerical factors** (ACMNA191) ***Understanding:*** ***describe patterns involving indices, connecting rules for linear relations and their graphs.*** ***Fluency:*** ***includes formulating, and modelling practical situations involving ratios, profit and loss, and areas and perimeters of common shapes.*** **Year 9:** - **Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems** (ACMNA208) - **Apply index laws** to numerical expressions with integer indices (ACMNA209) - **Extend and apply the index laws** to variables, using positive integer indices and the zero index (ACMNA212) - **Apply the distributive law to the expansion of algebraic expressions,** including binomials, and **collect like terms** where appropriate (ACMNA213) ***Understanding:*** ***describe the relationship between graphs and equations.*** ***Fluency:*** ***applying the index laws to expressions with integer indices.***

For this trial version of the Manual it has not been possible to include a similar mapping to the *National Numeracy Progression* as this is still in preparation. It is anticipated that this will be available in future versions.

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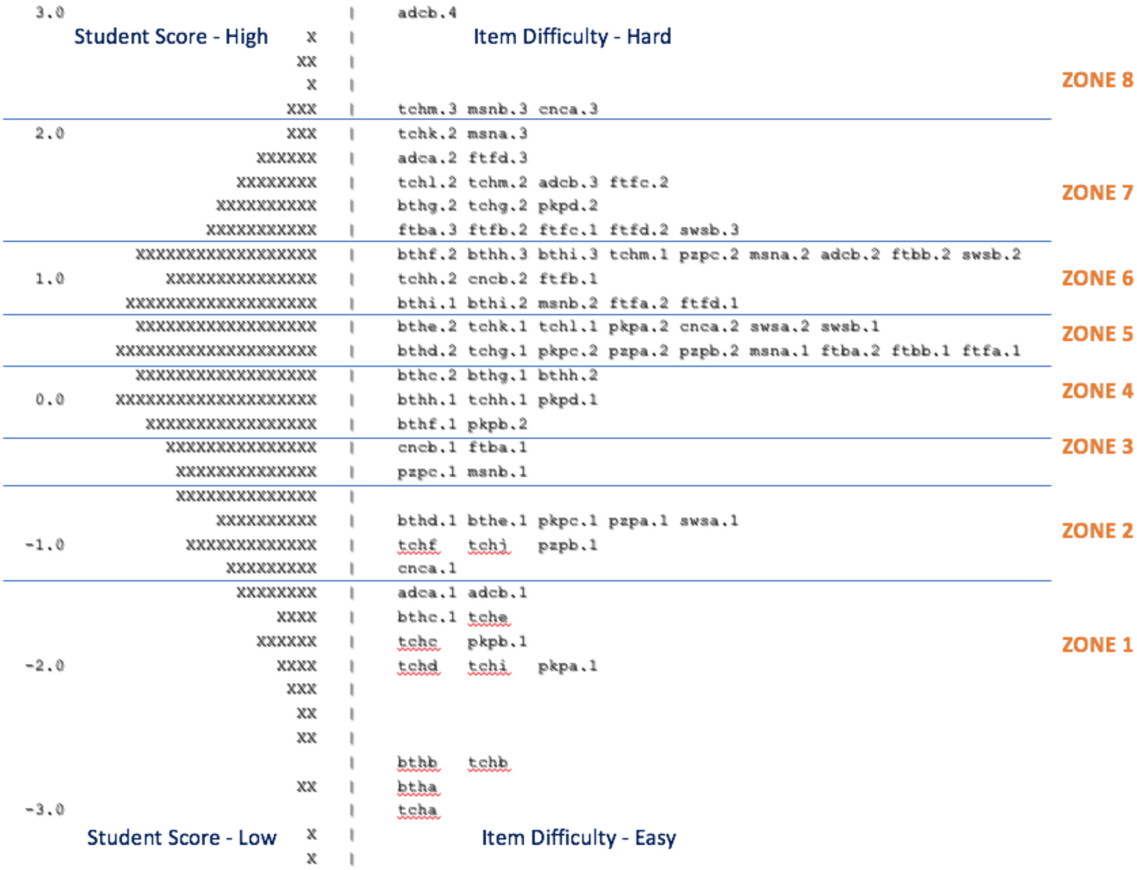
Appendix 1

The analysis of the SNMY student data produced a map that related student scores (low to high) to item difficulties (easy to hard) as shown below. The item analysis facilitated the identification of an eight-level learning progression for multiplicative thinking that described a range of behaviours from additive, count all strategies (Zone 1) to the sophisticated use of proportional reasoning (Zone 8) with multiplicative thinking not evident on a consistent basis until Zone 4. It also supported the development of Zone-based teaching advice referred to as the *Learning and Assessment Framework for Multiplicative thinking* or LAF for short.



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Variable Map SNMY Project 2006

As individual students are located on the same scale at the point where they have a 50% chance of successfully completing the items at that level of difficulty, the advice for each Zone is presented in terms of what needs to be consolidated and established and what needs to be introduced and developed to scaffold students' progression to the next Zone.