Reframing Mathematical Futures II

Algebraic Reasoning Learning Progression - Teaching Advice

Three big ideas that underpin algebraic reasoning have been identified as Pattern and Function, Equivalence, and Generalisation. These big ideas are not discrete but are intertwined. The Algebraic Learning Progression and the associated Teaching Advice have been structured around these three big ideas that are briefly described here.

- **Pattern and Function**

Pattern and Function is strongly based on noticing structure. Included within this big idea is noticing the structure of patterns, including number patterns and the structure of arithmetic. Within Pattern and Function, is the identification of relationships and inverse relationships, as well as the identification of variables and constants. Being able to move flexibly between multiple representations is an important aspect, as are the ideas of continuous and discrete functions, rates of change and families of functions.

- **Equivalence**

This big idea has at its core the notion of balance or balancing, including both known and unknown terms. Equivalence includes being able to understand the meaning of the equals sign, using comparative language, transforming and identifying equivalent expressions, especially in the solution of equations and inequations. The development of relational thinking is an important aspect. This includes using the four operations to produce equivalent numerical expressions, by recognising and creating equivalent sums, differences, products and quotients, and using equivalence to simplify calculations. In using symbolic expressions, students know how to combine two (or more) equivalent symbolic expressions retaining equivalence.

- **Generalisation**

Generalisation strongly connects arithmetic and algebraic thinking by focussing on similarities and differences. It relies on the noticing of structure. Generalisation includes moving from the specific to the general and moving from the general to the specific. This includes a variety of representations, such as models, words, pictures, graphs and symbols. The structure of arithmetic, number and algebraic laws, patterns, functions and equivalence situations may offer opportunities to develop generalisation. The meaningful use of mathematical language and symbols may also be developed through the process of generalisation.

Prepared by Lorraine Day, Marj Horne & Max Stephens, RMFII Research Team, July 2018

ZONE 1 Description	Teaching Implications
Pattern and Function *Can continue simple patterns, but likely to build them additively.* *Interprets data in tables as simple counts* *Cannot yet connect the scale of a graph to information provided in a table to result in a graphical representation.* ***Equivalence*** *Cannot yet see relationships between entities (e.g. 1 bag = 3 balls).* ***Generalisation*** *Reluctant to attempt any explanation.* *Unable to notice the structural form of patterns.*	**Consolidate and Establish** ➢ Work on repeating and growing patterns with concrete materials (e.g. by copying, continuing (in both directions), identifying repeating elements, describing in own words, completing, translating patterns into different medium, creating patterns, making tables of data from patterns, and putting pattern data on simple dot graphs) ➢ Create own axes for plotting data ➢ Make connections with number patterns ➢ Use balances to work on equivalence (*nRich: Getting the Balance*) Use balance situations with multiple groups on each side (e.g. 1 bag plus 6 balls balances with 2 bags and 2 balls - needs very light bags with appropriate balls inside) ➢ Present strategies for developing descriptions and justifications Orally > Pictures > Written in words **Introduce and Develop** ➢ Work with number patterns with an emphasis on multiples ➢ Use function machines with input output tables using simple linear rules, initially one operation then two have students create own rules ➢ Identify multiple representations of functions – words, tables, graphs and symbols as they relate to the words (*Maths300: 4-Arm Shapes*) ➢ Move between different representations with functions. ➢ Work with number sentences with unknowns in different places and describe the relationships between the numbers on each side of the equality. (*nRich: Number Balance 4725*) ➢ Ask students to explain/justify answers in all activities, encouraging the use of pictures as well (*Maths300: Lining Up*) ➢ Focus on composite units to encourage recognition of multiplicative situations. ➢ Work with more complex growing and number patterns, paying attention to structure. (*ReSolve: Number Sequences 5676*) ➢ Identify patterns in tables both across and down tables of values. ➢ Use input/output to generate possible connecting rules/relationships ➢ Move between different representations of pictures/tables/graphs. ➢ Explore balance scales for equivalence including halves (*nRich: Balance of Halves 5677*) ➢ Use compensating changes to work with number sentences with unknowns in different places. ➢ Ask the question what changes and what stays the same? In all of the situations mentioned above.

ZONE 2 Description	Teaching Implications
Pattern and Function *Can recognise some multiples – e.g. 60 as a multiple of 6.* *Recognises more complex patterns, but still working additively. Has an intuitive appreciation of pattern.* *Can work with simple scales and transfer from a table of values to a graph (two representations).* *Coordinate relationships are point-wise (e.g., ABBUY1.1).* ***Equivalence*** *Recognises relationships like 6 more/6 less, while not necessarily recognising equivalence.* ***Generalisation*** *Can sometimes respond correctly but with little/no reasoning.*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Include symbolic representation in the different representations, words/pictures/tables/graphs/symbols. ➢ Tell the story from a graph and make graphs from stories. ➢ Present strategies for developing descriptions – Orally > Pictures > Written in words. ➢ Use comparative language answering the question what is the same and what is different, in different contexts. ➢ Go from a generalisation to a number of specific examples and from some specific examples to a generalisation. (*ReSolve: Odds and Evens*) ➢ Describe and represent multiplicative relationships in multiple ways. ➢ Notice the structure of concrete, contextualised growing patterns, identifying what changes and what stays the same. (*Maths300: Painted Rods, nRich: Sticky Triangles 88*) ➢ Look for patterns across and down a table of values. ➢ Use the function machine idea of input output to create a non-sequential table of values and generate a rule in words and symbols. ➢ Use balance and number scale work to identify equivalent expressions. (*nRich: Shape Times Shape 5714*) ➢ Reason with equivalence and compensation strategies. ➢ Relational thinking. (*ReSolve: Chess the King*) ➢ Identify strategies for solving simple proportion problems. ➢ Encourage explanations using words, diagrams, graphs, symbols and examples within a variety of contexts. (*Maths300: Match Triangles*)

ZONE 3 Description	Teaching Implications
Pattern and Function *Can recognise simple rules and express them in words and symbols, but likely to focus on additive relationships. Can extend patterns but may not be able to explain them, although is working more comfortably with composite units. Recognise relationships (e.g. ARLT1.1) but are not yet using or applying them to solve problems efficiently (e.g. ARLT1.2). Can understand simple multiplicative relationships but cannot explain or apply beyond (e.g. ALEM1). Unable to make comparisons beyond a concrete situation (e.g. ABBUY2.1)* ***Equivalence*** *Can recognise some equivalent relationships, but may not be able to fully explain them.* ***Generalisation*** *Is beginning to work with simple generalisations like odd + odd = even but justifies with a single example.*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Recognise and work with more complex patterns including squares and other multiplicative ideas, and fractions. ➢ Move between different forms of function and in real contexts and informally identify the meaning of the range and domain. ➢ Identify what varies and what is constant in relation/function situations. ➢ Identify similarities and differences in relational thinking situations when using different operations and use missing numbers on both sides to generate generalisations. (*Maths300: Arithmagons, nRich: Super Shapes*) ➢ Move from specific examples to generalisations and generalisations to specific examples. ➢ Justify arguments and generalisation by using specific examples. Discuss how many examples you need before you are "sure". (*nRich: Magic Vs 6274*) ➢ Explore both complex additive and multiplicative relationships using tables, symbols, graphs and words and discuss the connections between them. ➢ Use appropriate representations of discrete and continuous situations informally within a context. Does it make sense to join the dots? In what other contexts would it make sense? (*Maths300: Garden Beds*) ➢ Work with domain and range informally within a context. Does it make sense to extrapolate? Is it appropriate to interpolate? ➢ Identify variables (what changes), recognise that a letter can stand for multiple numbers. ➢ Identify strategies for solving proportional reasoning problems involving ratios and rates. ➢ Recognise equivalent expressions by substituting numbers > generalisation. (*Maths 300: Garden Beds*) ➢ Work with 'proof' firstly by single example > proof by multiple examples > Proof in general sense. ➢ Record and describe situations in multiple ways (multiple representations).

ZONE 4 Description	Teaching Implications
Pattern and Function *Can recognise more complex additive relationships and apply them, may not be able to support with reasoning or necessarily identify the inherent multiplicative relationship. Can apply multiplicative thinking in input/output situations (e.g. ABBEQ 1.2) Beginning to work with ratios. Can recognise and interpret relevance of range from table and/or graphs (e.g. AHAB2.1). Able to recognise that variables (what changes) and constants (what stays the same) need to be treated differently. Beginning to recognise functional relationships (e.g. ALEM2.1).* ***Equivalence*** *Can recognise the impact of operations (e.g. subtraction and addition relationships behave differently.) (e.g. ARELS4.1). May attempt to 'solve' symbolic algebraic expressions by giving the variable a numerical value.* ***Generalisation*** *Able to work intelligently with composite units (what changes and what stays the same). Provides 'proof' by examples. Beginning to transition to abstraction by inserting a number for a pronumeral.*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Move to a different representation with simple functional representations such as input/output tables, graphs or symbols, particularly relating to continuous graphical situations. Identify non- integer points as well as interpolating and extrapolating values. ➢ Use simple real relationships to explore the shape of graphs – e.g. distance time relating to going for an actual walk. ➢ Relate the structure of a situation to the function generated. (*Maths300: Unseen Triangles*) ➢ Explore different strategies for maintaining equivalence. ➢ Use relational thinking ideas with subtraction and large numbers as well as with missing numbers on both sides to enable generalisation. ➢ Explain and justify using both words and mathematical text. ➢ In solving worded problems focus on the relationships between the quantities in the problem ➢ Investigate relationships containing number using concrete materials to see how patterns are structured. (*e.g. Max's matchstick problems, Mountain Range Challenge, Garden Beds*) ➢ Move from just recognising patterns to identifying relationships. (*Maths300: Heads and Legs*) ➢ Notice structure to form generalisations. ➢ Work with Function Machines both forwards and backwards to identify inverse operations. (*Maths300: What's My Rule*) ➢ Unpack equations with attention to their structure. ➢ Notice structure in relational thinking situations. ➢ Use thinking strings to see different structures. ➢ Identify variables and constants by recognising what changes and what stays the same. ➢ Use multiple embodiments to help move away from context. ➢ Explain and justify using mathematical text and diagrams in a variety of contexts.

ZONE 5 Description	Teaching Implications
Pattern and Function *Able to recognise and represent simple functional representations in input/output situations and as continuous graphical representations (e.g. ABBEQ3). Can interpolate value(s) in simple continuous relationships which implies an intuitive understanding of the shape of a graph (e.g. AHAB1.3).* *Recognises structural relationships. Has an appreciation of relationships but may need to use calculations (e.g. AHAB3.3). Assigns a number to a variable (e.g. ARELS7.1).* ***Equivalence*** *Is reasoning with more complex additive situations involving larger numbers and subtraction, but generally by example. Can derive a strategy that maintains equivalence, but cannot yet generalise (cannot understand need to maintain balance) (e.g. ARELS3.1).* ***Generalisation*** *Beginning to work with generalisations and using symbols abstractly (e.g. AMUST3). Can justify arguments using mathematical text. Beginning to generalise but still need to build on or connect to the known.*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Investigate relationships in real life contexts that involve ratio and other multiplicative relationships. ➢ Explore situations where a variable can be used multiple times (e.g. the perimeter of the long tiles) and express the findings in symbols. ➢ Interpret and create graphs intuitively as they relate to real life situations (e.g. distance from home against time on a trip to the shops, speed of a racing car as it goes around a given track). (*Maths300: Speed Graphs*) ➢ Explore inverses and maintaining equivalence, particularly additively. ➢ Generalise from multiple examples in arithmetic situations. ➢ Investigate questions that ask students to work backwards. (*Maths300: Backtracking, nRich: Think of Two Numbers 1170*) ➢ Explore questions for which there is not a sensible answer within the context. ➢ Move flexibly between multiple representations. ➢ Move from symbolic structure to simplest form by recognising equivalent expressions. (*ReSolve: Tens and Units*) ➢ Explore activities where lots of answers are possible – "now another one, now another one…" followed by generalising, and describing and justifying the rule. ➢ Scaffold the generalisation process. (*Maths300: Game of 31*)

ZONE 6 Description	Teaching Implications
Pattern and Function *Is able to describe and justify rules involving multiplicative relationships (e.g. ATRNS2.3). Can use ratios (e.g. ALEM3.1). Is beginning to work with symbols to express rules, but not necessarily in the simplest way. Is able to describe and generate rules when a prompt is present (ARELS3.2 compared with ARELS1.3).* ***Equivalence*** *Recognises the additive inverse and maintaining equivalence (e.g. ARELS3.2). Can understand and strengthen more complex relational reasoning situations.* ***Generalisation*** *Able to generalise simple arithmetic relationships (can see the general in the specific) (e.g. ABRT3.2).*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Explore meaningful situations where there is more than one relationship using multiple representations. (*nRich: How Much Can We Spend? 6650*) ➢ Explore multiplicative reasoning with direct and inverse variation and proportional reasoning. (*nRich: Burning Down 497*) ➢ Move between different representations of a function in a variety of contexts, including abstract ones. ➢ Explore generalisations about number relationships where justification (leading to proof) can be developed (e.g. if the sum of the digits is divisible by 9 the number is divisible by 9). (*Maths300: Consecutive Sums, nRich: Please Explain 1006*) ➢ Explore simple examples of operations research involving constraints. ➢ Appreciate the relationships between functions and different ways of expressing general functional notation. ➢ Examine the impact of gradients and constants on linear graphs anduse different letters, rather than just common ones like y = mx + c. (*Maths300: Algebra Walk, ReSolve: Think of a Number – Linear Equations*) ➢ Explain what is happening to someone else or in another real situation using multiple representations. ➢ Formalise the recognition and description of number patterns and use algebraic text. (*Maths300: Four Piles Problem*) ➢ Solve algebraic problems involving multiplicative relationships and develop a greater range of strategies. (*nRich: Please Explain*) ➢ Simplify algebraic expressions, manipulate algebraic text and move flexibly between equivalent forms. (*Maths300: Algebra Charts, ReSolve: Working with Algebra, Addition Chain*) ➢ Express generalisations in different ways, using multiple representations.

ZONE 7 Description	Teaching Implications
Pattern and Function *Can interpolate data and work with range and accept some degree of uncertainty (e.g. AHAB3.3). Is able to qualify statements to appreciate a final relationship (e.g. BBUY2.3). Can identify the relationships between two functions and identify cross-over points (e.g. ABBUY2.3). Can describe multiplicative relationships using two variables expressed abstractly (proportional reasoning) (e.g. ALEM3.2). Is able to move flexibly between multiple representations.* ***Equivalence*** *Can reason based on the relationships on either side of an equals sign (e.g. ARELS1.3, ARLT1.3).* ***Generalisation*** *Can describe in words a generalised relationship derived from an equation (e.g. ARELS7.2).*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Solve problem situations with general solutions, using mathematical argument to justify solutions. (*nRich: Square Mean 357*) ➢ Make generalisations about classes of numbers, expressions or functions and justify them (for example use technology to draw a series of a family of curves and generalise their properties). (*Maths300: Algebra Walk, Pick's Rule*) ➢ Become familiar with simple non-linear relationships (*Maths300: Painted Cubes, Staircases, nRich: Cubes Within Cubes Revisited 2824, Christmas Chocolates 6675*) ➢ Recognise that within a limited domain, multiple representations may be possible. ➢ Explain the assumptions which are likely to justify a particular representation or model and discuss the domain over which these assumptions are likely to apply.

ZONE 8 Description	Teaching Implications
Pattern and Function *Can visualise the form and structure of a function at least graphically from a real context.* ***Equivalence*** *Is able to relate the use of abstract symbols to another idea, such as perimeter and is beginning to use the simplest form to do so (e.g. ATILP3).* ***Generalisation*** *Is able to reason based on a relationship between numbers (e.g. ACONS2.2). Can successfully work through problems that have multiple steps.*	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Explore families of functions and make generalisations about their properties. (*Maths300: Painted Cubes, Triangles and Colours*) ➢ Investigate general forms of curves and their relationship to properties. (*Maths 300: Gradient Functions*) ➢ Investigate, interpret and represent direct and inverse proportion relationships numerically, graphically and algebraically and use these to solve problems. ➢ Investigate problems involving the intersection of linear and non-linear functions algebraically, numerically and graphically, applying these to realistic contexts. ➢ Investigate and interpret graphs of piecewise functions and other relations that model practical contexts. ➢ Explore the connections between algebraic and graphical representations and transformations of different families of functions. ➢ Investigate the use of exponential and logarithmic scales to model situations involving growth and decay.