Reframing Mathematical Futures II

Algebraic Reasoning Learning Progression - Teaching Advice

Three big ideas that underpin algebraic reasoning have been identified as Pattern and Function, Equivalence, and Generalisation. These big ideas are not discrete, but are intertwined. The Algebraic Learning Progression and the associated Teaching Advice have been structured around these three big ideas that are briefly described here.

- **Pattern and Function**

Pattern and Function is strongly based on noticing structure. Included within this big idea is noticing the structure of patterns, including number patterns and the structure of arithmetic. Within Pattern and Function, is the identification of relationships and inverse relationships, as well as the identification of variables and constants. Being able to move flexibly between multiple representations is an important aspect, as are the ideas of continuous and discrete functions, rates of change and families of functions.

- **Equivalence**

This big idea has at its core the notion of balance or balancing, including both known and unknown terms. Equivalence includes being able to understand the meaning of the equals sign, using comparative language, transforming and identifying equivalent expressions, especially in the solution of equations and inequations. The development of relational thinking is an important aspect. This includes using the four operations to produce equivalent numerical expressions, by recognising and creating equivalent sums, differences, products and quotients, and using equivalence to simplify calculations. In using symbolic expressions, students know how to combine two (or more) equivalent symbolic expressions retaining equivalence.

- **Generalisation**

Generalisation strongly connects arithmetic and algebraic thinking by focussing on similarities and differences. It relies on the noticing of structure. Generalisation includes moving from the specific to the general and moving from the general to the specific. This includes a variety of representations, such as models, words, pictures, graphs and symbols. The structure of arithmetic, number and algebraic laws, patterns, functions and equivalence situations may offer opportunities to develop generalisation. The meaningful use of mathematical language and symbols may also be developed through the process of generalisation.

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ZONE 1 Description	Teaching Implications
Can continue simple patterns, but is likely to build them additively. Reasoning is confined to specific incidences and numerical examples of simple physical situations. Arithmetic thinking is used. Abstraction and generalisation not evident at this stage.	**Consolidate and Establish** ➢ Work on repeating and growing patterns with concrete materials Copying Continuing (in both directions) Identifying repeating elements Describing in own words Completing Translating patterns into different medium Creating patterns Making tables of data from patterns Putting pattern data on simple dot graphs ➢ Create own axes for plotting data ➢ Make connections with number patterns ➢ Use balances to work on equivalence (*nRich: Getting the Balance*) Explore balance and recording what balances Use balance situations with multiple groups on each side – e.g. 1 bag plus 6 balls balances with 2 bags and 2 balls (need very light bags with appropriate balls inside) ➢ Present strategies for developing descriptions and justifications Orally > Pictures > Written in words **Introduce and Develop** ➢ Work with number patterns with an emphasis on multiples ➢ Use function machines with input output tables using simple linear rules, initially one operation then two have students create own rules ➢ Identify multiple representations of functions – words, tables, graphs and symbols as they relate to the words (*Maths300: 4-Arm Shapes*) ➢ Move between different representations with functions. ➢ Work with number sentences with unknowns in different places and describe the relationships between the numbers on each side of the equality. (*nRich: Number Balance 4725*) ➢ Ask students to explain/justify answers in all activities, encouraging the use of pictures as well (*Maths300: Lining Up*) ➢ Focus on composite units to encourage recognition of multiplicative situations. ➢ Work with more complex growing and number patterns, paying attention to structure. (*ReSolve: Number Sequences 5676*) ➢ Identify patterns in tables both across and down tables of values. ➢ Use input/output to generate possible connecting rules/relationships ➢ Move between different representations of pictures/tables/graphs. ➢ Explore balance scales for equivalence including halves (*nRich: Balance of Halves 5677*) ➢ Use compensating changes to work with number sentences with unknowns in different places. ➢ Ask the question what changes and what stays the same? In all of the situations mentioned above.

ZONE 2 Description	Teaching Implications
Beginning to recognise patterns and relationships and conjecture about these. Able to identify numbers that vary and numbers that stay the same. Engages with the context, but arithmetic reasoning, typically based on calculations, is still being used. Recognises some multiples and some relationships like 6 more/6 less, while not necessarily recognising equivalence. Can work with simple scales and transfer from a table of values to a graph.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Include symbolic representation in the different representations words/pictures/tables/graphs/symbols. ➢ Tell the story from a graph and make graphs from stories. ➢ Present strategies for developing descriptions – Orally > Pictures > Written in words. ➢ Use comparative language answering the question what is the same and what is different, in different contexts. ➢ Go from a generalisation to a number of specific examples and from some specific examples to a generalisation. (*ReSolve: Odds and Evens*) ➢ Describe and represent multiplicative relationships in multiple ways. ➢ Notice the structure of concrete, contextualised growing patterns, identifying what changes and what stays the same. (*Maths300: Painted Rods, nRich: Sticky Triangles 88*) ➢ Look for patterns across and down a table of values. ➢ Use the function machine idea of input output to create a non-sequential table of values and generate a rule in words and symbols. ➢ Use balance and number scale work to identify equivalent expressions. (*nRich: Shape Times Shape 5714*) ➢ Reason with equivalence and compensation strategies. ➢ Relational thinking. (*ReSolve: Chess the King*) ➢ Identify strategies for solving simple proportion problems. ➢ Encourage explanations using words, diagrams, graphs, symbols and examples within a variety of contexts. (*Maths300: Match Triangles*)

ZONE 3 Description	Teaching Implications
Beginning to use symbolic expression and elementary reasoning. While still using arithmetic approaches there is evidence of relational reasoning with the numbers and providing some explanation. Beginning to recognise simple multiplicative relationships. There is some evidence of co-ordination of two ideas. Explanation and justification is limited. Algebraic expressions are used rather than equations. Beginning to recognise equivalent relationships. Can explain simple generalisations by telling stories, manipulating materials and very simple use of symbolic language.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Recognise and work with more complex patterns including squares and other multiplicative ideas, and fractions. ➢ Move between different forms of function and in real contexts and informally identify the meaning of the range and domain. ➢ Identify what varies and what is constant in relation/function situations. ➢ Identify similarities and differences in relational thinking situations when using different operations and use missing numbers on both sides to generate generalisations. (*Maths300: Arithmagons, nRich: Super Shapes*) ➢ Move from specific examples to generalisations and generalisations to specific examples. ➢ Justify arguments and generalisation by using specific examples. Discuss how many examples you need before you are "sure". (*nRich: Magic Vs 6274*) ➢ Explore both complex additive and multiplicative relationships using tables, symbols, graphs and words and discuss the connections between them. ➢ Use appropriate representations of discrete and continuous situations informally within a context. Does it make sense to join the dots? In what other contexts would it make sense? (*Maths300: Garden Beds*) ➢ Work with domain and range informally within a context. Does it make sense to extrapolate? Is it appropriate to interpolate? ➢ Identify variables (what changes), recognise that a letter can stand for multiple numbers. ➢ Identify strategies for solving proportional reasoning problems involving ratios and rates. ➢ Recognise equivalent expressions by substituting numbers > generalisation. (*Maths 300: Garden Beds*) ➢ Work with 'proof' firstly by single example > proof by multiple examples > Proof in general sense. ➢ Record and describe situations in multiple ways (multiple representations).

ZONE 4 Description	Teaching Implications
Beginning to work multiplicatively and simultaneously co-ordinate variables, although still uses specific examples to convince. Able to reason and generalise in simple situations. Can recognise and interpret the relevance of range from table and/or graphs and to recognise functional relationships. When faced with more complex algebraic situations is unable to use the full range of explanation or handle all of the information simultaneously. Beginning to transition to abstraction by inserting a number for a pronumeral.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Move to a different representation with simple functional representations such as input/output tables, graphs or symbols, particularly relating to continuous graphical situations. Identify non- integer points as well as interpolating and extrapolating values. ➢ Use simple real relationships to explore the shape of graphs – e.g. distance time relating to going for an actual walk. ➢ Relate the structure of a situation to the function generated. (*Maths300: Unseen Triangles*) ➢ Explore different strategies for maintaining equivalence. ➢ Use relational thinking ideas with subtraction and large numbers as well as with missing numbers on both sides to enable generalisation. ➢ Explain and justify using both words and mathematical text. ➢ In solving worded problems focus on the relationships between the quantities in the problem ➢ Investigate relationships containing number using concrete materials to see how patterns are structured. (*e.g. Max's matchstick problems, Mountain Range Challenge, Garden Beds*) ➢ Move from just recognising patterns to identifying relationships. (*Maths300: Heads and Legs*) ➢ Notice structure to form generalisations. ➢ Work with Function Machines both forwards and backwards to identify inverse operations. (*Maths300: What's My Rule*) ➢ Unpack equations with attention to their structure. ➢ Notice structure in relational thinking situations. ➢ Use thinking strings to see different structures. ➢ Identify variables and constants by recognising what changes and what stays the same. ➢ Use multiple embodiments to help move away from context. ➢ Explain and justify using mathematical text and diagrams in a variety of contexts.

ZONE 5 Description	Teaching Implications
Able to use multiplicative reasoning in simple situations. Can reason with more complex additive situations involving larger numbers and subtraction but usually by examples. Has moved from algebraic expressions to using equations. Can derive a strategy that maintains equivalence, but cannot yet generalise the situation. Able to use symbols to express rules. Can follow, compare and explain rules for linking successive terms in a sequence. Recognises and represents simple functional representations. Can justify an argument using mathematical text. Beginning to generalise using words or using some symbolic generalisations in simple situations, usually building on in context.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Investigate relationships in real life contexts that involve ratio and other multiplicative relationships. ➢ Explore situations where a variable can be used multiple times (e.g. the perimeter of the long tiles) and express the findings in symbols. ➢ Interpret and create graphs intuitively as they relate to real life situations (e.g. distance from home against time on a trip to the shops, speed of a racing car as it goes around a given track). (*Maths300: Speed Graphs*) ➢ Explore inverses and maintaining equivalence, particularly additively. ➢ Generalise from multiple examples in arithmetic situations. ➢ Investigate questions that ask students to work backwards. (*Maths300: Backtracking, nRich: Think of Two Numbers 1170*) ➢ Explore questions for which there is not a sensible answer within the context. ➢ Move flexibly between multiple representations. ➢ Move from symbolic structure to simplest form by recognising equivalent expressions. (*ReSolve: Tens and Units*) ➢ Explore activities where lots of answers are possible – "now another one, now another one…" followed by generalising, and describing and justifying the rule. ➢ Scaffold the generalisation process. (*Maths300: Game of 31*)

ZONE 6 Description	Teaching Implications
Can use and interpret basic algebraic conventions to represent situations involving a variable quantity. Beginning to explain using logical language and to use if … then reasoning. Uses symbolic language but the need for simplification is still being developed. Able to generalise arithmetic relationships with justification, including simple multiplicative relationships, but are often still context bound. Can show why several expressions are equivalent, typically employing numerical (non-symbolic) justifications.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Explore meaningful situations where there is more than one relationship using multiple representations. (*nRich: How Much Can We Spend? 6650*) ➢ Explore multiplicative reasoning with direct and inverse variation and proportional reasoning. (*nRich: Burning Down 497*) ➢ Move between different representations of a function in a variety of contexts, including abstract ones. ➢ Explore generalisations about number relationships where justification (leading to proof) can be developed (e.g. if the sum of the digits is divisible by 9 the number is divisible by 9). (*Maths300: Consecutive Sums, nRich: Please Explain 1006*) ➢ Explore simple examples of operations research involving constraints. ➢ Appreciate the relationships between functions and different ways of expressing general functional notation. ➢ Examine the impact of gradients and constants on linear graphs anduse different letters, rather than just common ones like y = mx + c. (*Maths300: Algebra Walk, ReSolve: Think of a Number – Linear Equations*) ➢ Explain what is happening to someone else or in another real situation using multiple representations. ➢ Formalise the recognition and description of number patterns and use algebraic text. (*Maths300: Four Piles Problem*) ➢ Solve algebraic problems involving multiplicative relationships and develop a greater range of strategies. (*nRich: Please Explain*) ➢ Simplify algebraic expressions, manipulate algebraic text and move flexibly between equivalent forms. (*Maths300: Algebra Charts, ReSolve: Working with Algebra, Addition Chain*) ➢ Express generalisations in different ways, using multiple representations.

ZONE 7 Description	Teaching Implications
Is able to use and interpret algebraic conventions for representing generality and relationships between variables. Beginning to use sound logical reasoning with appropriate reasoning language (e.g. if … then, must) evident. There is more co-ordination of multiplicative thinking and the associated language to notice algebraic structure. Can recognise and use the relationships between multiple entities and connections between and within different representations. Is able to establish and describe equivalence explaining relationships using the distributive property and the inverses of addition and multiplication. Can generalise quite complex situations and in more direct situations is beginning to use simplest form.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Solve problem situations with general solutions, using mathematical argument to justify solutions. (*nRich: Square Mean 357*) ➢ Make generalisations about classes of numbers, expressions or functions and justify them (for example use technology to draw a series of a family of curves and generalise their properties). (*Maths300: Algebra Walk, Pick's Rule*) ➢ Become familiar with simple non-linear relationships (*Maths300: Painted Cubes, Staircases, nRich: Cubes Within Cubes Revisited 2824, Christmas Chocolates 6675*) ➢ Recognise that within a limited domain multiple representations may be possible. ➢ Explain the assumptions which are likely to justify a particular representation or model and discuss the domain over which these assumptions are likely to apply.

ZONE 8 Description	Teaching Implications
Is able to combine a facility with symbolic representation and an understanding of algebraic concepts to represent and explain mathematical situations. Explanations are sophisticated using logical thought and the language of reasoning. Can use multiple representations in a co-ordinated manner to solve, analyse, convince and conclude. Can visualise the form and structure of a function, at least graphically, from a real context. Is able to work in a context free environment using symbolic language and treat algebraic expressions (e.g. 3x + 2) as single entities. Can generalise more complex situations. Is able to establish and describe equivalence involving the four operations explaining relationships in symbolic terms. Can use abstract symbols to solve problems in context with multiple steps.	**Consolidate and Establish** ➢ What was Introduced and Developed in previous Zone **Introduce and Develop** ➢ Explore families of functions and make generalisations about their properties. (*Maths300: Painted Cubes, Triangles and Colours*) ➢ Investigate general forms of curves and their relationship to properties. (*Maths 300: Gradient Functions*)