GROWING Mathematically

Multiplicative Thinking

Teacher's Manual

Background reading

(Trial Version)

Growing Mathematically is a project funded by the Australian Federal Government. It builds on materials that were developed as part of the RMIT Reframing Mathematical Futures II and the Scaffolding Numeracy in the Middle Years (SNMY) projects. SNMY content is currently hosted by the Victorian Department of Education.

GROWING MATHEMATICALLY: Multiplicative Thinking

(DRAFT) TEACHER MANUAL

Supporting a targeted teaching approach to multiplicative thinking in the middle years based on an evidenced-based learning progression

This resource has been produced by the Australian Association of Mathematics Teachers (AAMT), in collaboration with Emerita Professor Di Siemon of RMIT and her colleagues with funding from the Commonwealth Government of Australia.

The aim of the Manual is to add value to the existing formative assessment materials for multiplicative thinking developed by the *Scaffolding Numeracy in the Middle Years Project* (Siemon, Breed, Dole, Izard, & Virgona, 2006)[1].

The update has been made possible by the results of the *Reframing Mathematical Futures* (2013–2018) projects that explored the efficacy of using the SNMY materials in secondary schools alongside the development of similar formative assessment materials for algebraic, geometrical and statistical reasoning (Siemon, Callingham, Day, Horne, Seah, Stevens, & Watson, 2018).

CONTENTS

- **What is MT?** Definition of multiplicative thinking
- **Why is MT important?** Brief description of the SNMY and RMFII projects, data to show MT the issue in the middle years and targeted teaching works!
- **Supporting the development of multiplicative thinking** – the SNMY formative assessment materials
- **Targeted teaching –** description and evidence to show that TT works
- **Instructions –** how to administer and mark assessment options
- **Student work samples –** to help interpret scoring rubrics
- **Learning Assessment Framework for Multiplicative Thinking** - learning progression, teaching advice, and link to related resources
- **Case Studies** – examples to show how schools have implemented a TT approach
- **ACARA Mapping –** alignment between MT Learning Progression, the *Australian Curriculum: Mathematics*, and the National Numeracy Continuum

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1 https://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/assessment/Pages/scaffoldnum.aspx

- **References/Further Reading** – links to relevant research papers, wider reading (to be completed)
- **Research Basis** – Appendices (to be added)

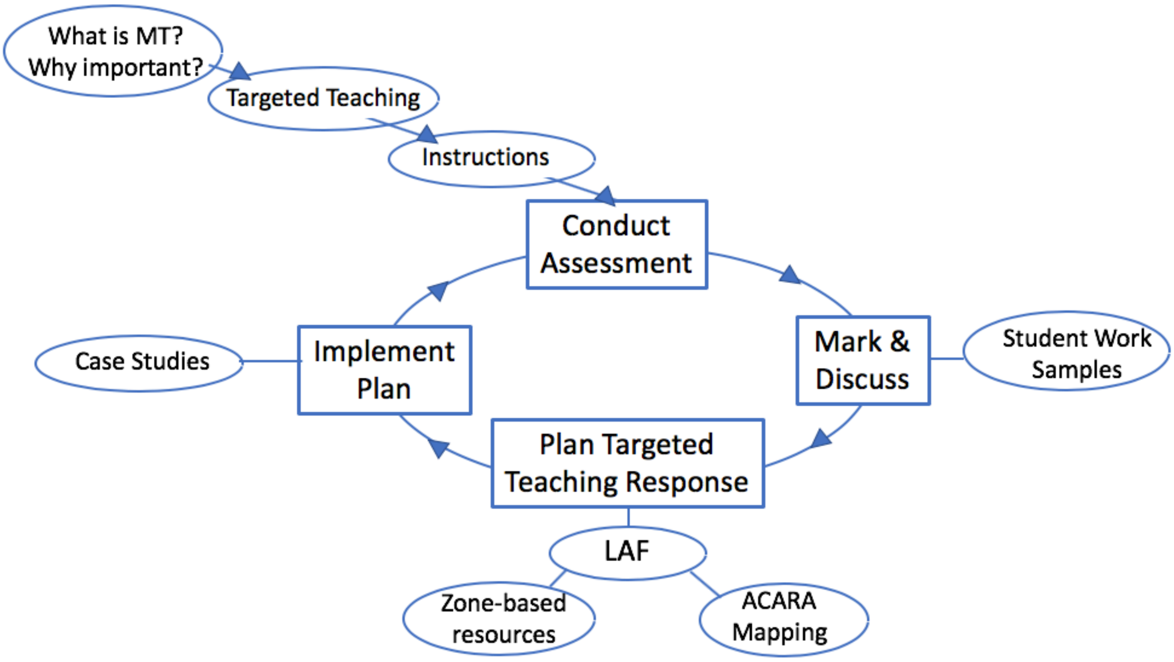

Figure 1. A conceptual map of the Growing Mathematically – Multiplicative Thinking Resource.

What is multiplicative thinking?

Multiplicative thinking involves recognising and working with relationships between quantities. Although some aspects of multiplicative thinking are available to young children, multiplicative thinking is substantially more complex than additive thinking and may take many years to achieve (Lamon, 2012; Vergnaud, 1983). This is because multiplicative thinking is concerned with processes such as replicating, shrinking, enlarging, and exponentiating that are fundamentally more complex than the more obvious processes of aggregation and disaggregation associated with additive thinking and the use of whole numbers (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015).

Multiplicative thinking is qualitatively different to additive thinking. It is evident when students:

- work flexibly and confidently with an extended range of numbers (i.e. larger whole numbers, fractions decimals, per cent, and ratios);
- solve problems involving multiplication and division, including direct and indirect proportion using strategies appropriate to the task; and
- explain and communicate their reasoning in a variety of ways (e.g. words, diagrams, symbolic expressions, and written algorithms. (Siemon, Breed, & Virgona, 2005).

In short, where additive thinking involves the aggregation or disaggregation of collections (e.g., $634 + $478 or finding the difference between 82 kg and 67 kg), multiplicative thinking involves reasoning with relationships between quantities, for example,

- 3 bags of wool per sheep, 5 sheep, how many bags of wool?,
- At an average speed of 85 km/hour, how long will it take to travel 367 km?.

Additive problems generally involve one measure space (e.g., dollars or kilograms) while multiplicative problems generally involve working with two (or more) measure spaces (e.g. bags of wool, number of sheep) and a relationship between the two (i.e. 3 bags of wool per sheep).

Because simple multiplicative problems such as '24 strawberry plants per row, 17 rows, how many strawberry plants?' can be solved additively using repeated addition or by using a learnt algorithm and known facts, it can be difficult to determine whether or not a student is thinking multiplicatively. Where this becomes apparent is where the problems involve larger whole numbers, fractions, decimals, per cent or ratios, and/or more complex relationships between quantities. For example, the following problems will generally provoke a range of strategies, not all of which are multiplicative

> A muffin recipe uses 2/3 cup of milk to make 12 muffins. How many muffins can be made with 6 cups of milk?
>
> A small business owner wants to offer a further 20% discount on her summer clothing range, but she needs to ensure she covers the wholesale price. The wholesale price of a summer top was $73. If the original price of the top was $139 and it was currently on sale for 30%, can she offer a 20% discount on the already discounted price?
>
> Mobile phone covers are offered in 5 different sizes, 3 different styles, and 14 different colours. How many different phone covers need to be ordered to have 3 of each type in stock?

Sam said that doubling the dimensions of the garden box would double the volume. Is he correct? Use as much mathematics as you can to justify your conclusion.

If it takes 3 men 24 hours to paint a house, how long will it take 2 men to paint the house?

A wildlife officer estimated that there were 73 koalas in one forest reserve of 328 hectares and 62 in another forest reserve of 263 hectares. Which forest reserve provided more space for each koala?

Why is multiplicative thinking important?

Multiplicative thinking is crucial to success in school mathematics. It underpins nearly all of the topics considered in the middle years and beyond (see Siemon, 2013) and it is fundamental to careers in science, technology, engineering and mathematics (STEM).

Multiplicative thinking is needed to support efficient solutions to more difficult problems involving multiplication and division, fractions, decimal fractions, ratio, rates and percentage, and to solve proportional reasoning problems as they arise in algebra, geometry, measurement, statistics, and probability.

However, Australian research suggests that at least 25% and up to 55% of students in Year 8 do not have access to this critical capacity (Siemon, Breed, Dole, Izard, & Virgona, 2006; Siemon, 2013, 2016, 2019; Siemon, Banks, & Prasad, 2018).

A large-scale study involving just under 7000 Victorian students in Years 5 to 9 found that there was a seven-year range in student mathematics achievement in each year level, which was almost entirely due to the extent to which students had access to multiplicative thinking (Siemon & Virgona, 2001). More recent studies involving up to 32 secondary schools across Australia have confirmed that access to multiplicative thinking remains the reason for the significant difference in student mathematics achievement in Years 7 to 9 (e.g., Siemon, 2013, 2016, 2019; Siemon, Banks, & Prassad, 2018).

Lack of access to multiplicative thinking helps explain the reported decline in the performance of Australian students on international assessments of mathematics (e.g. Thompson, De Bortoli, Underwood, & Schmid, 2019) and the significant decline in the proportion of Year 12 students undertaking the more advanced mathematics courses. But the research also reveals significant inequalities in that students from low socioeconomic communities are far more likely to be represented in the 45 to 55% range of students not having access to multiplicative thinking than students from higher socioeconomic backgrounds, who are more likely to be represented in the 25 to 35% range. This situation is untenable where the fastest growing employment opportunities require some form of STEM qualification.

What can be done to support the development of multiplicative thinking?

Identifying and building on what students know in relation to important mathematics is widely regarded as the key to improving learning outcomes (e.g., Black & Wiliam, 1998; Goss, Hunter, Romanes & Parsonage, 2015; Masters, 2013; Timperley, 2009; Wiliam, 2011).

Moreover, where teachers are supported to identify and interpret student learning needs, they are more informed about where to start teaching, and better able to scaffold their students' mathematical learning (Callingham, 2010; Clarke, 2001; Siemon, 2016).

In response to the initial research project that identified multiplicative thinking as the source of the seven-year range in mathematics achievement (Siemon & Virgona, 2001), the *Scaffolding Numeracy in the Middle Years* (SNMY) project (2004-2006) used rich tasks and Rasch modelling to investigate the development of multiplicative thinking in just over 3200 students in Years 4 to 8 (Siemon & Breed, 2006; Siemon, Breed, Dole, Izard, & Virgona, 2006). The following resources were developed as a result of the project.

- **A Learning and Assessment Framework for Multiplicative Thinking (LAF)** that comprises an evidenced-based, eight-level learning progression for multiplicative thinking that describes a range of behaviours from additive, count all strategies (Zone 1) to the sophisticated use of proportional reasoning (Zone 8) with multiplicative thinking not evident on a consistent basis until Zone 4. Detailed targeted teaching advice that provides information on what needs to be consolidated and established at each Zone as well as what needs to be introduced and developed to scaffold student learning to the next Zone is also provided (see below)
- **Two validated assessment options** consisting of an extended task and five or six shorter tasks each of which contain two or more items. Partial credit scoring rubrics that value core knowledge, the ability to apply that knowledge, and the capacity to explain and justify are provided as well as two Raw Score Translators that map student scores to the one of the Zones of the learning progression
- **Additional Zone-based resources** were also provided in the form of learning plans and authentic tasks.

The SNMY project also demonstrated that teaching targeted to individual student learning needs can make a significant difference. For example, Breed (2011) undertook a doctoral study as part of the SNMY project. Nine Year 6 students identified in Zone 1 of the LAF in 2004 participated in an 18-week intervention in mid 2005. The students worked with the teacher in small groups using manipulatives, games, discussion and weekly written reflections using the LAF as a guide. When re-assessed three months after the intervention, all nine students shifted at least 4 zones with the majority shifting five Zones to be age and grade appropriate.

Targeted teaching

Targeted teaching is a form of differentiation that is specifically concerned with students' learning needs in relation to a small number of **'big ideas'** in Number, in this case, multiplicative thinking, without which their progress in school mathematics will be seriously impacted (Siemon, 2006; 2017; Siemon, Bleckly, & Neal, 2012).

Targeted teaching is based on the premise that there are three key processes in involved in improving a student's mathematics learning:

- understanding where the learner is right now,
- understanding where the learner needs to be, and
- understanding how to get there (Wiliam, 2013)

Targeted teaching requires:

- **access to accurate information** about what each student is able to do (i.e., reliable, evidence-based eliciting tools)
- **interpretations of student behaviour** in terms of the key steps in the development of important mathematical ideas and strategies
- a **commitment to acting on the evidence** to inform both in-the-moment and future teaching (i.e., to use the evidence obtained to better target the learning needs of all students)
- an **expanded repertoire of teaching approaches** that accommodate and nurture discourse, help uncover and explore students' ideas in constructive ways, and ensure all students can participate in and contribute to the enterprise; and
- **flexibility** to spend time with those who need it most (Siemon, 2006)

The targeted teaching cycle for multiplicative thinking using the SNMY resources is shown in Figure 2 below.

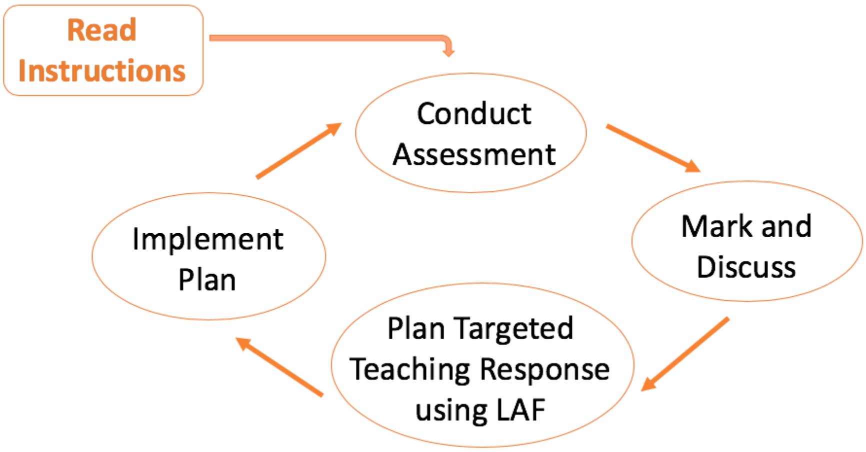

Figure 2. The targeted teaching cycle.

Targeting multiplicative thinking works

Targeted teaching is not easy but where implemented effectively, it can make a significant difference to student mathematics learning outcomes.

2006 – Overall medium to large effect sizes[2] (in the range 0.45 to 0.75 or more) were found across the SNMY research schools (17 primary, 3 secondary) compared to small to medium effect sizes (in the range of 0.2 to 0.5) in the reference schools (Siemon, Breed, Dole, Izard, & Virgona, 2006).

2011 – Breed (2011) reported shifts of up to four Zones as a result of a targeted, 18-week intervention based on the Learning and Assessment Framework for Multiplicative Thinking.

2013 – The results of the *Reframing Mathematical Futures - Priority (RMF-P)* project demonstrated the efficacy of adopting a targeted teaching approach to multiplicative thinking using the SNMY materials in Years 7 to 9 (e.g., Siemon, 2016; Siemon, Banks, & Prassad, 2018).

[2] An effect size of 0.4 or greater is considered to represent an improvement above what might otherwise be expected (Hattie, 2012).

The average effect size across the 28 schools was 0.64, however, individual school results ranged from 0.4 to 1.2 (see Case Study, 1. p. 22).

2015 –The Grattan Institute report on *Targeted Teaching: How better use of data can improve student learning* (Goss, Hunter, Romanes, & Hunter, 2015) presents the general case for formative assessment and three case studies to showcase the benefits of adopting a targeted teaching approach

2016 – Of the 10 schools that used the SNMY materials in Years 7 and 8 in the context of the *Reframing Mathematical Futures II* project (e.g., Siemon, 2019; Siemon, Banks, & Prassad, 2018), the average effect size was 0.47. Again, individual school results ranged considerably, but four schools achieved effect sizes well in excess of 1.0 (e.g. see Case Study 2, p. 23).

A range of factors were nominated by the teachers involved in the RMF projects as reasons for the differential results. These included the extent to which the targeted teaching approach was endorsed and practically supported by school leadership, the availability of planning and professional learning time, access to appropriate spaces and resources, and the varying levels of staff 'buy in'. However, the teachers also reported that working together to moderate and discuss student responses was one of "the best professional development opportunities" they had experienced (Siemon, 2019).